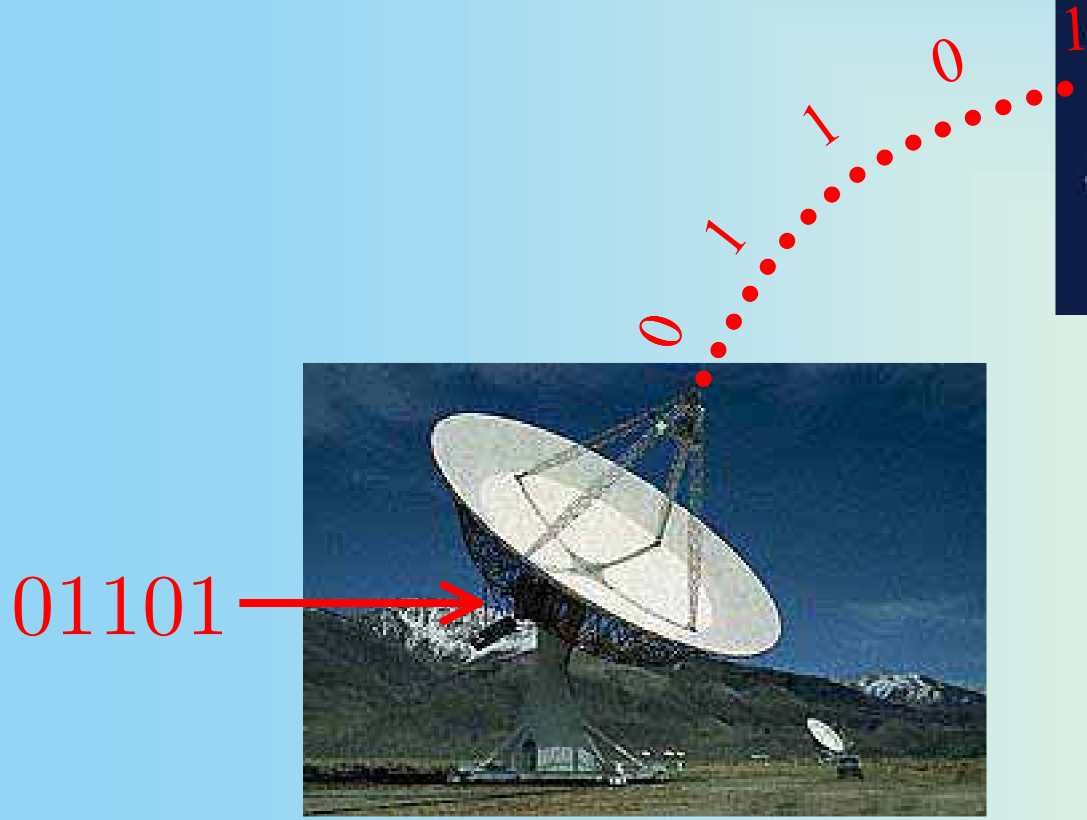
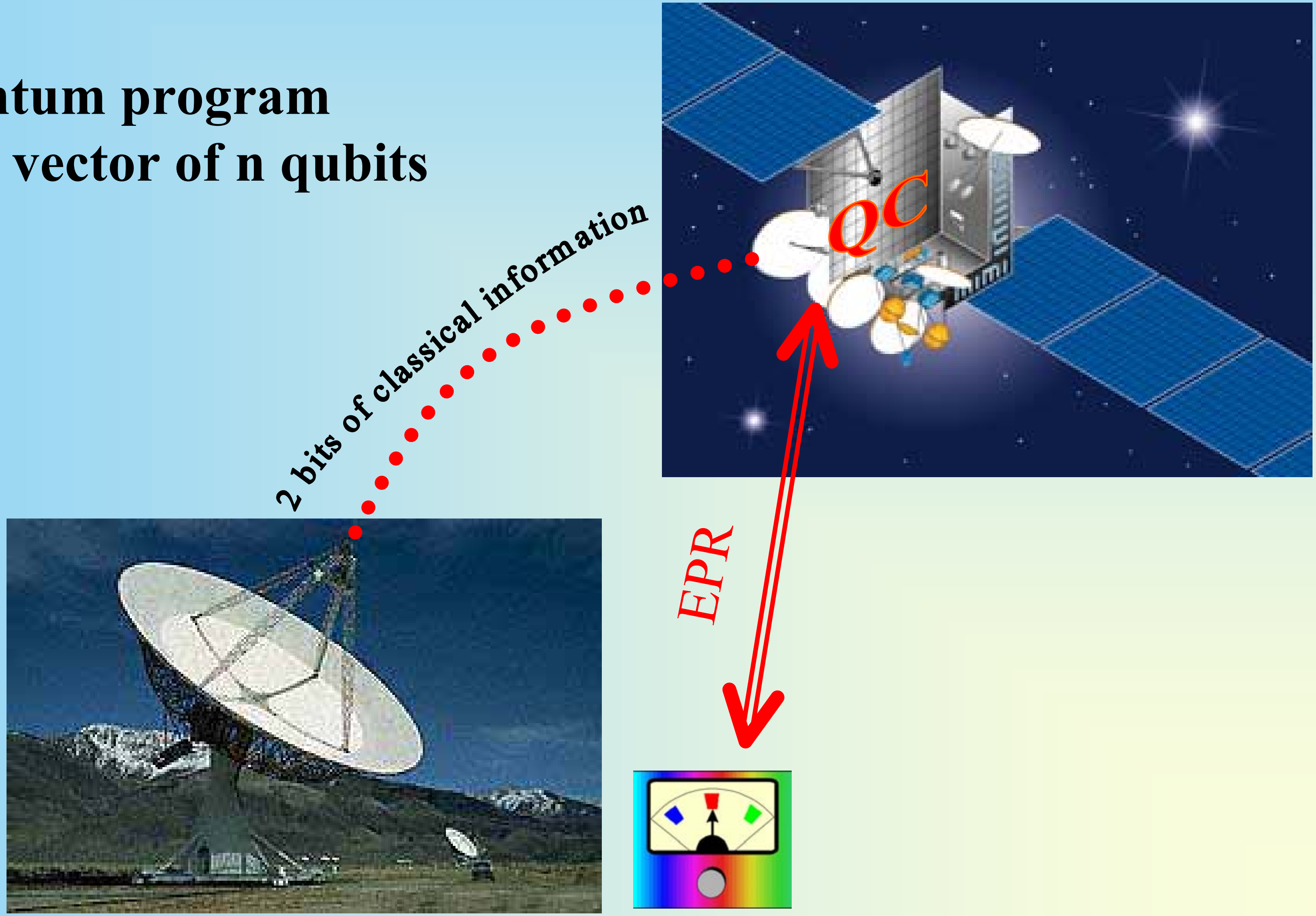
Motivation: Classical Program

Classical program : sequence of bits



Motivation: Quantum Program

Quantum program
State vector of n qubits



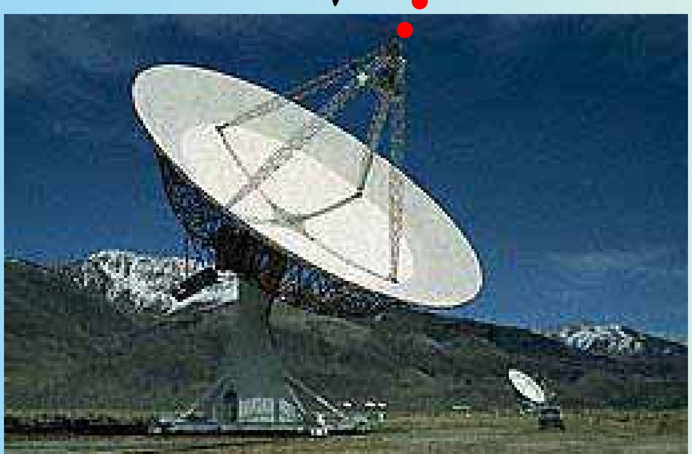
$|\Xi\rangle$

Motivation: Quantum Program

Quantum program
State vector of n qubits

2 bits of classical information

EPR



Universality ?

Research Center for Quantum Information
Bratislava, Slovakia

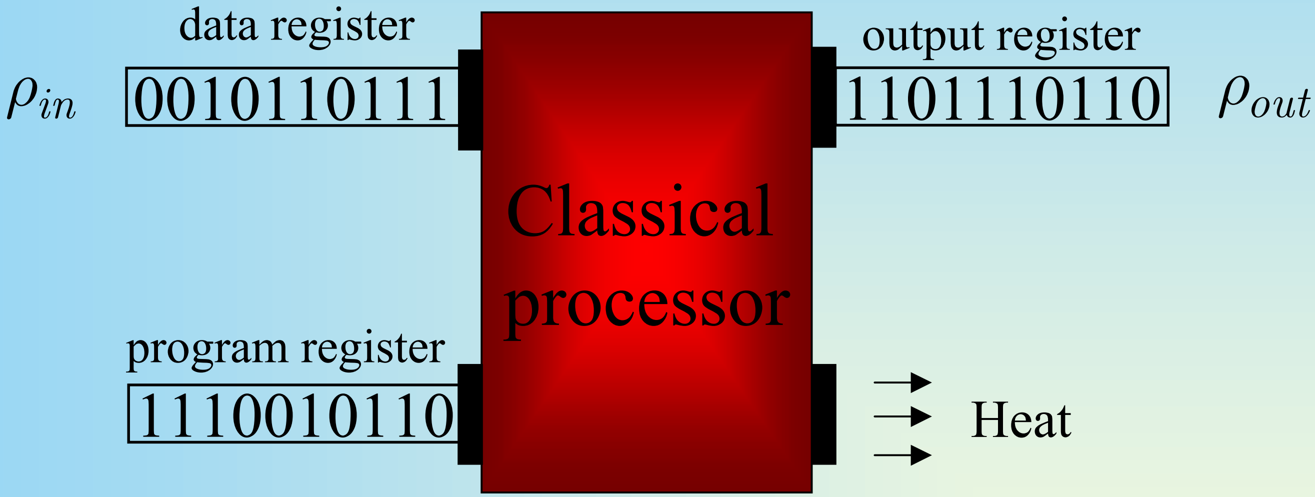
Programmable Quantum Processors

Vladimír Bužek
Mark Hillery
Mário Ziman

QUBITS $\rightarrow$ QGATES

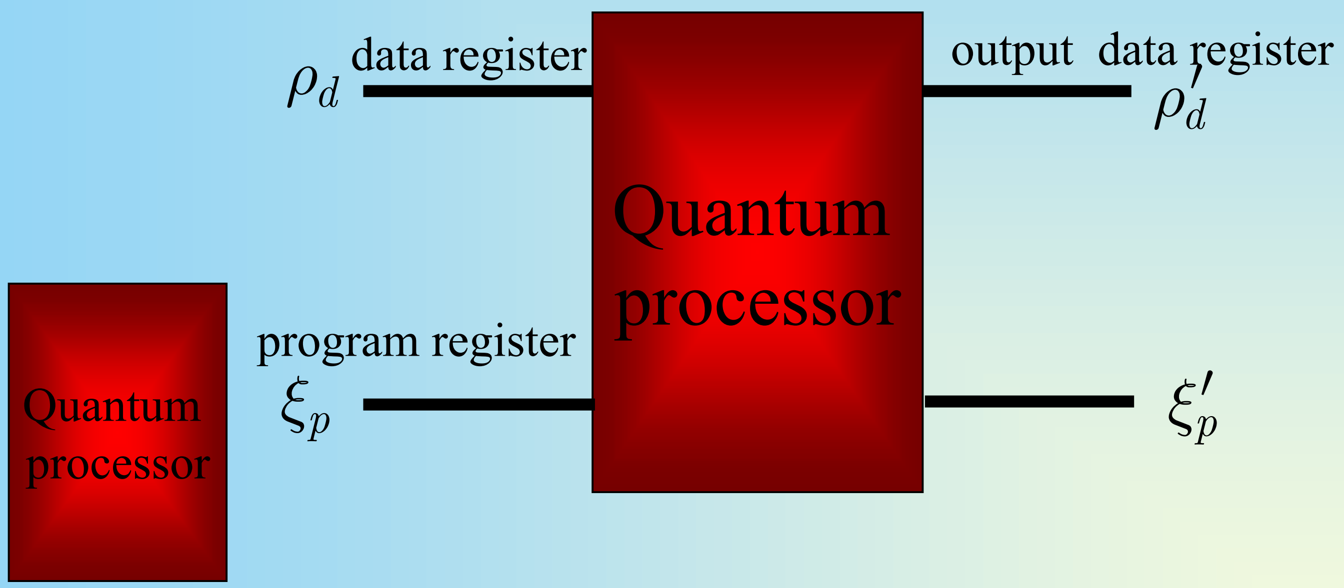
16 September, 2002, Dublin

Model of Classical Processor



$$\boxed{\rho_{out} = T[\rho_{in}]}$$

Quantum Processor



Quantum processor – fixed unitary transformation U_{dp}

$\mathcal{H}_d$ – data system, $S(\mathcal{H}_d)$ – data states

$\mathcal{H}_p$ – program system, $S(\mathcal{H}_p)$ – program states

Two Scenarios

- **Measurement-based strategy** - estimate the state of program

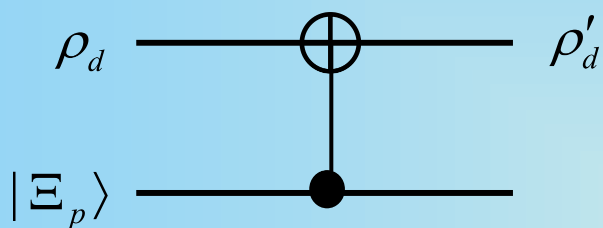
$$F = \frac{N+1}{N+d}$$

- **Quantum strategy** – use the quantum program register

  conditional (probabilistic) processors

  unconditional processors

C-NOT as Unconditional Quantum Processor



$$\mathrm{CNOT}\ |\psi\rangle|0\rangle = |\psi\rangle|0\rangle$$
$$\mathrm{CNOT}\ |\psi\rangle|1\rangle = \sigma_x|\psi\rangle\otimes|1\rangle$$

- program state $|0\rangle \Rightarrow \mathbf{1}$ implemented, i.e. $\rho_d \rightarrow \rho'_d = \rho_d$
- program state $|1\rangle \Rightarrow \sigma_x$ implemented, i.e. $\rho_d \rightarrow \rho'_d = \sigma_x\rho_d\sigma_x$
- general pure state $|\Xi_p\rangle = \alpha|0\rangle_p + \beta|1\rangle_p \Rightarrow \rho_d \mapsto \rho'_d = |\alpha|^2\rho_d + |\beta|^2\sigma_x\rho_d\sigma_x$
- unital operation, since $\Phi[\mathbf{1}] = |\alpha|^2\mathbf{1} + |\beta|^2\sigma_x\mathbf{1}\sigma_x = \mathbf{1}$
- program state is **2-d** and we can apply **2** unitary operations

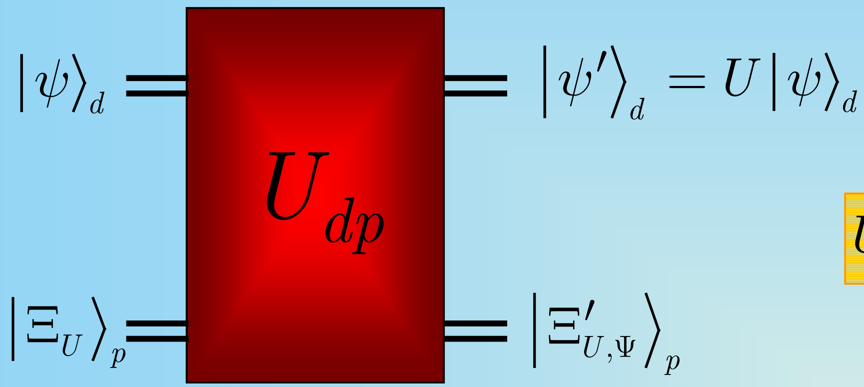
Question (Nielsen & Chuang)

Is it possible to build a *universal* programmable quantum gate array which take as input a quantum state specifying a quantum program and a data register to which the unitary operation is applied ?

on a qubit an ∞ number of operations can be performed

M.A.Nielsen & I.L.Chuang, *Phys. Rev. Lett* 79, 321 (1997)

Theorem



$$U_{dp}\left(|\psi\rangle_d \otimes |\Xi_U\rangle_p\right) = (U|\psi\rangle) \otimes |\Xi'_{U,\psi}\rangle$$

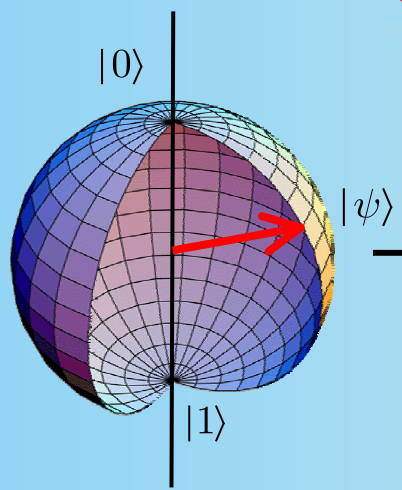
- **no universal deterministic quantum array of finite extent can be realized**
- **on the other hand – a program register with *d* dimensions can be used to implement *d* unitary operations by performing an appropriate sequence of controlled unitary operations**

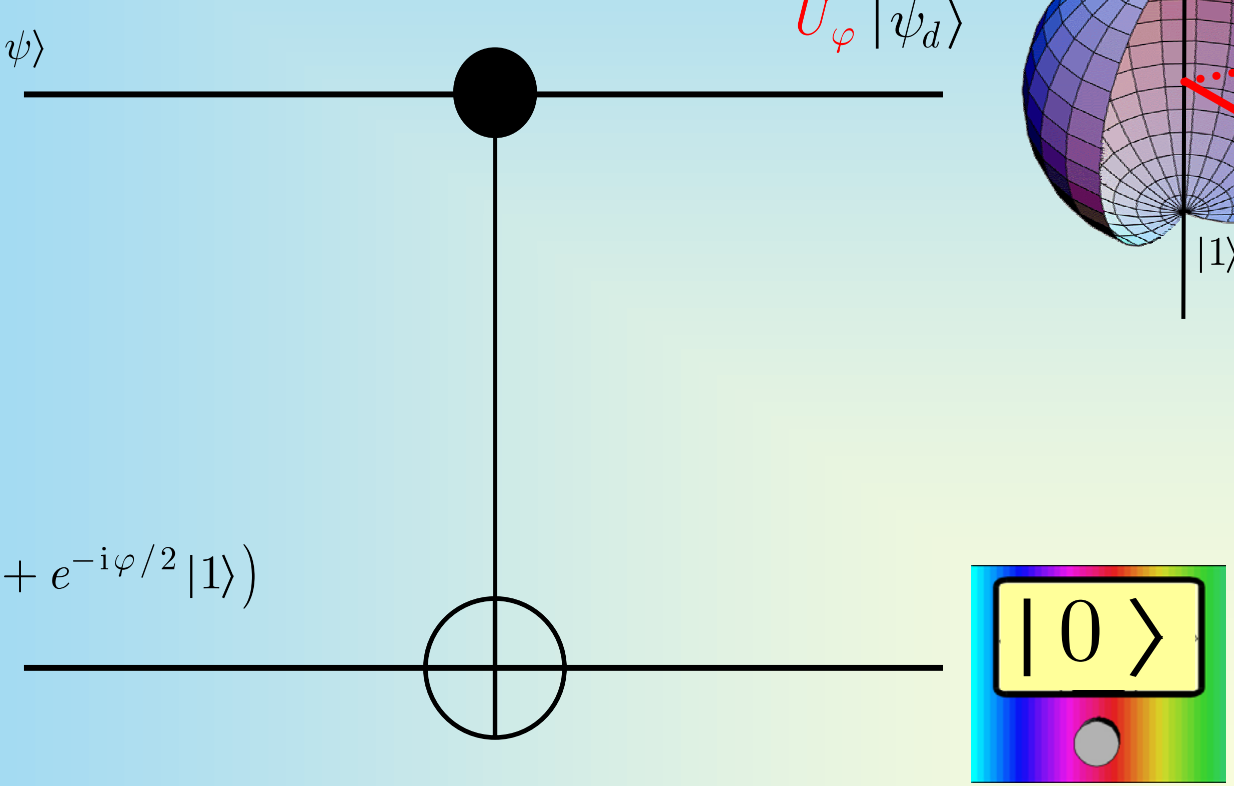
$$|\text{answer}\rangle = |\text{NO}\rangle|\text{deterministic}\rangle + |\text{YES}\rangle|\text{probabilistic}\rangle$$

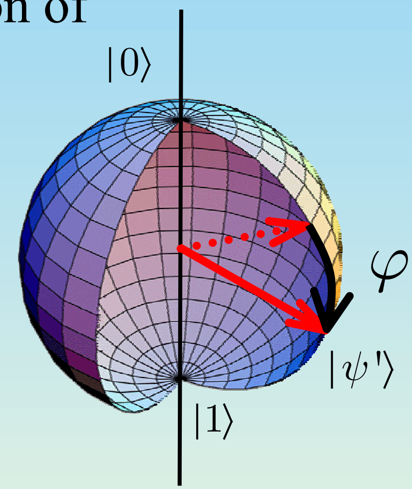
C-NOT as Probabilistic Quantum Processor

- Vidal & Cirac – probabilistic implementation of

$$U_\varphi = e^{-i\frac{\varphi}{2}\sigma_z}$$





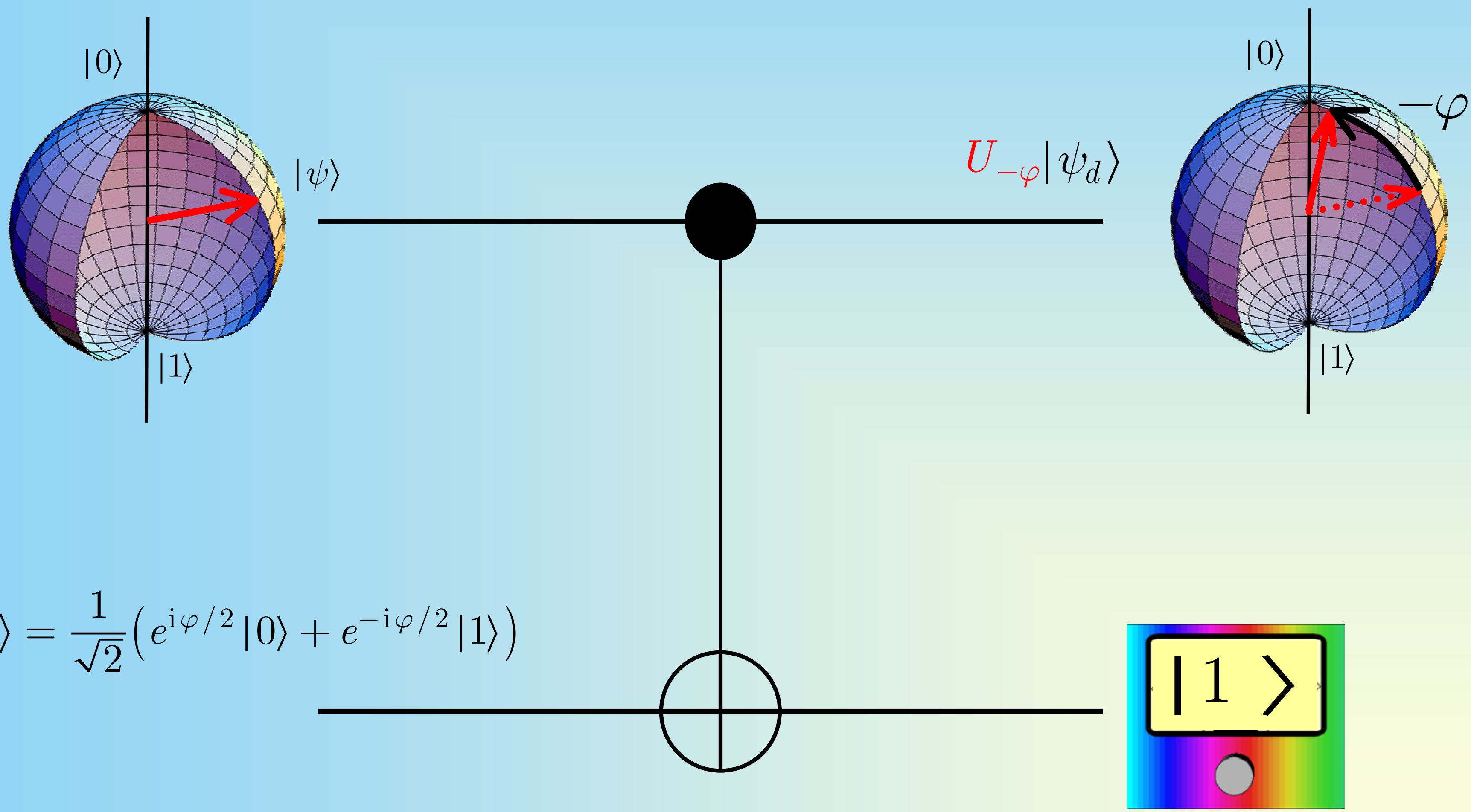


$$|\varphi\rangle = \frac{1}{\sqrt{2}}\left(e^{i\varphi/2}|0\rangle + e^{-i\varphi/2}|1\rangle\right)$$

G.Vidal and J.I.Cirac, Los Alamos arXiv *quant-ph/0012067* (2000)

G.Vidal, L.mesanes, and J.I.Cirac, Los Alamos arXiv *quant-ph/0102037* (2001).

C-NOT as Probabilistic Quantum Processor

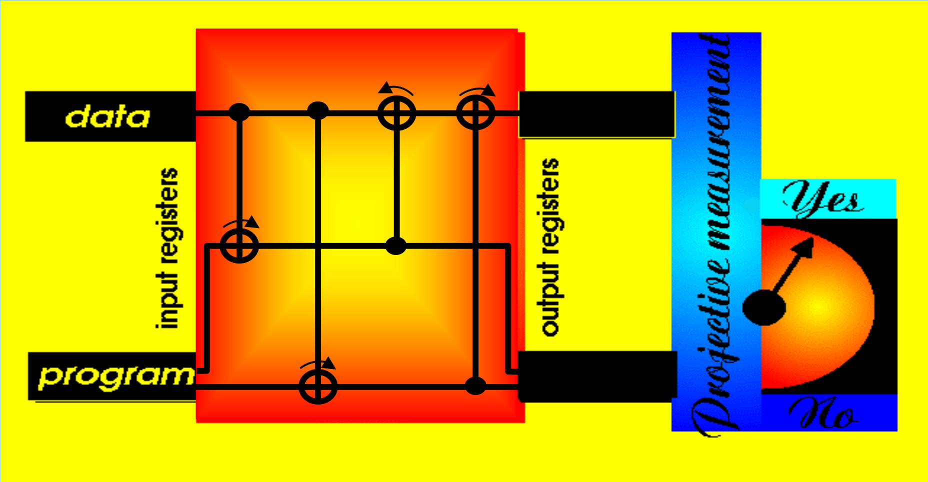


$$|\varphi\rangle = \frac{1}{\sqrt{2}}\left(e^{\mathrm{i}\varphi/2}|0\rangle + e^{-\mathrm{i}\varphi/2}|1\rangle\right)$$

Correction of the error – new run of the processor with $|2\varphi\rangle$

Universal Probabilistic Processor

ρ_d



$$\rho_d' = A\rho_d A^\dagger$$

$$A = \sum_k \alpha_k U_k$$

$$|\xi_p\rangle = \sum_k \alpha_k |k\rangle$$

$$|M\rangle = \frac{1}{D}\sum_k |k\rangle$$

- implementation of any unitary transformation
- BUT! only with probability and $P_{\text{success}} = (\dim \mathcal{H}_p)^{-1}$

•Example (Quantum information distributor)

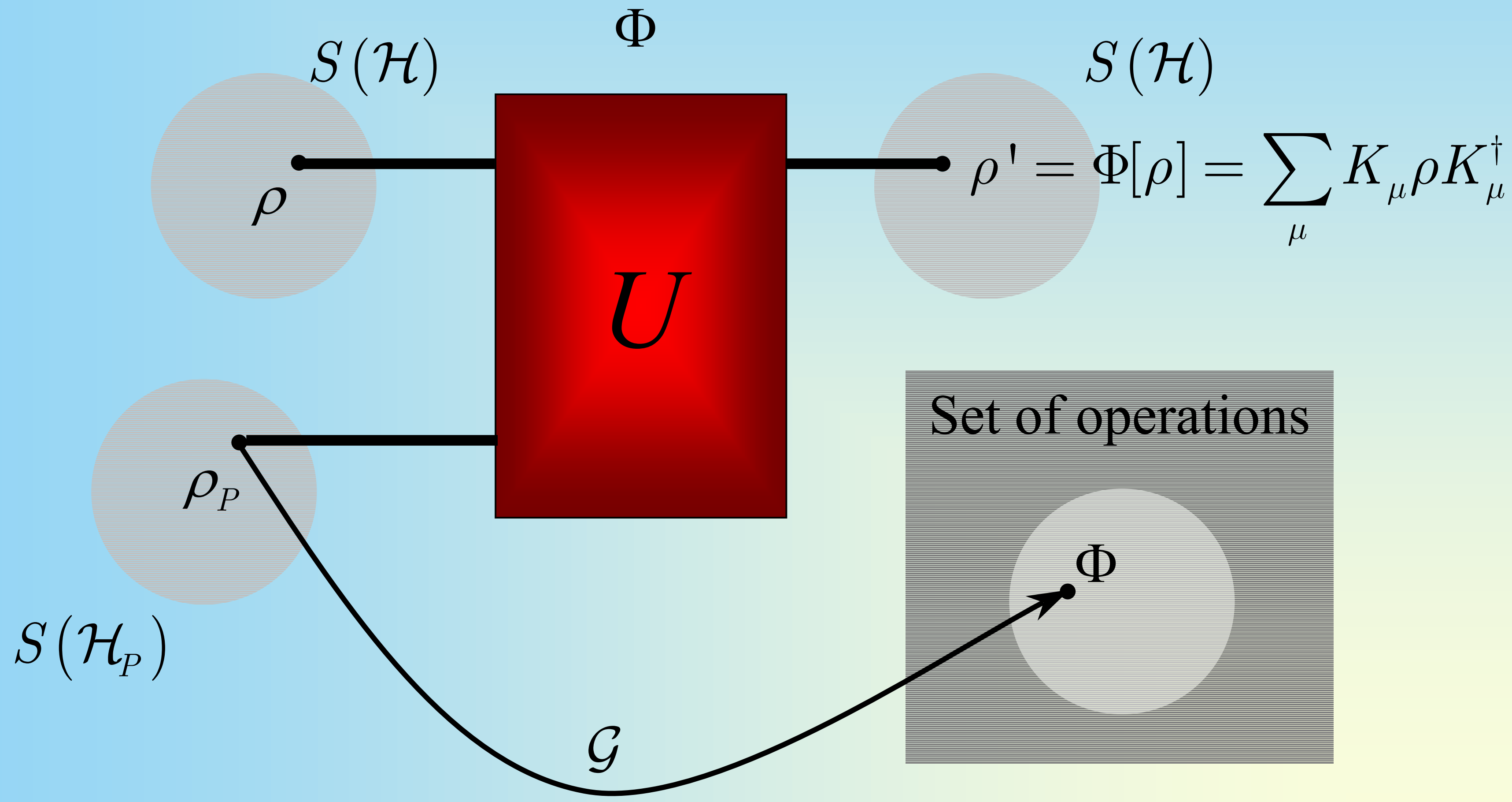
$\mathcal{H}_p = \mathcal{H}_d \otimes \mathcal{H}_d$; $D = \dim \mathcal{H}_d$

$\{|k\rangle\}_k = \{|k'\rangle\}_k$ ... basis of maximally entangled states of $\mathcal{H}_p$

$\{U_k\}_k$ ... set of D^2 unitaries forming an operator basis, $\mathbf{Tr}\, U_k^\dagger U_l = D\delta_{kl}$

- subset of unitary transformations – higher probability

Implementation of Maps via Unconditional Quantum Processors



Mathematical Description of Quantum Processors

- **definition of U_{dp} via "Kraus operators"** $A_{kl} := {}_p\langle l | U_{dp} | k \rangle_p$

$$U_{dp}\left(|\psi\rangle_d \otimes |k\rangle_p \right) = \sum_l \left(A_{kl} |\psi\rangle_d \right) \otimes |l\rangle_p$$

- **normalization condition** $\sum_l A_{k_1 l}^{\dagger} A_{k_2 l} = \delta_{k_1 k_2} \mathbf{1}_d$
- **induced quantum operation** $\rho_d \mapsto \rho_d' = \Phi_k[\rho_d] = \sum_l A_{kl} \rho_d A_{kl}^{\dagger}$
- **general pure program state** $|\Xi\rangle_p = \sum_k \alpha_k |k\rangle_p$

$$\rho_d \mapsto \rho_d' = \Phi_{\Xi}[\rho_d] = \sum_l A_l(\Xi) \rho_d A_l^{\dagger}(\Xi)$$

$$A_l(\Xi) = {}_p\langle l | U_{dp} | \Xi \rangle_p = \sum_k \alpha_k A_{kl}$$

- **can be generalized for mixed program states**

Classes of Processors

- Induced mapping

  $\mathcal{G}$: program states $\rightarrow$ set of operations

$$\xi_p \rightarrow \mathcal{G}\left(\xi_p\right) \equiv \Phi_\xi \quad ; \quad \Phi_\xi : S(\mathcal{H}_d) \rightarrow S(\mathcal{H}_d)$$

- Clases of processors

  1. U processors – implement unitary operations
  2. covariant processors
  3. injective processors – each ξ_p encodes different operation Φ_ξ

U processors

$$U_{dp}\left(|\psi\rangle_d \otimes |k\rangle_p\right) := \left(U_k|\psi\rangle_d \otimes |k'\rangle_p\right)$$

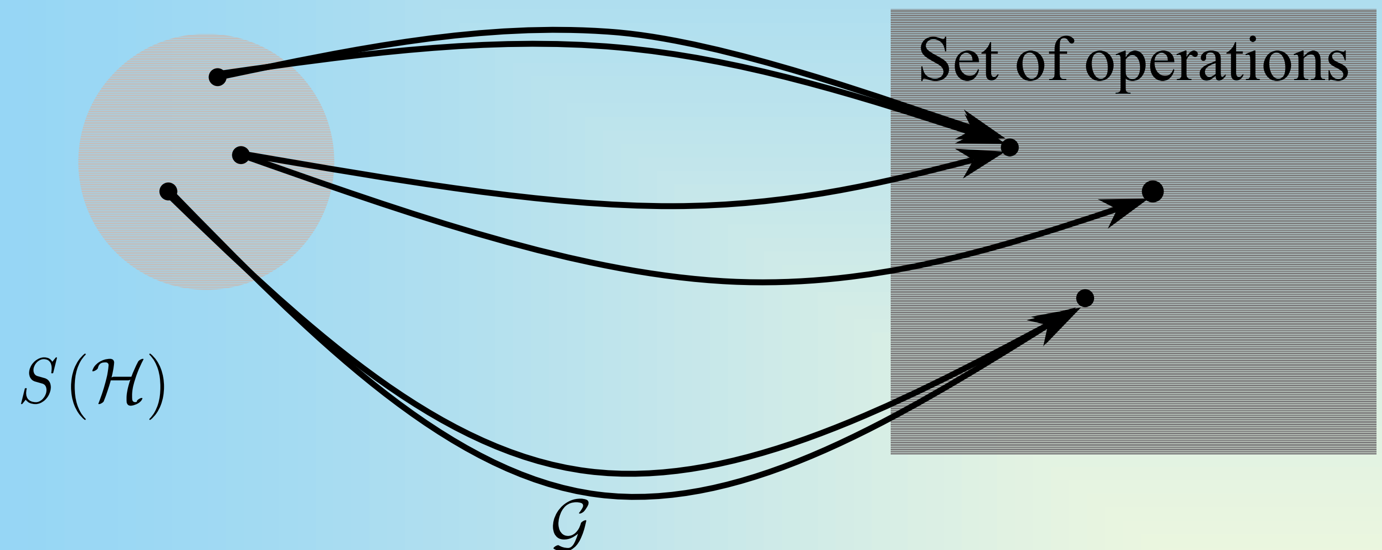
the most general processor that implements $N = \dim\,\mathcal{H}_p$ different unitaries

elementary programs $|k\rangle_p$ realize unitary transformation U_k

general program state ξ_p implements unital operations, i.e. $\Phi[\mathbf{1}] = \mathbf{1}$

Injective Processors

the induced mapping $\mathcal{G}$ is injective, i.e. $\forall$ pair $\Xi_{1p} \neq \Xi_{2p} \Rightarrow \Phi_1 \neq \Phi_2$



Does such processor exist? YES partial swap processors:

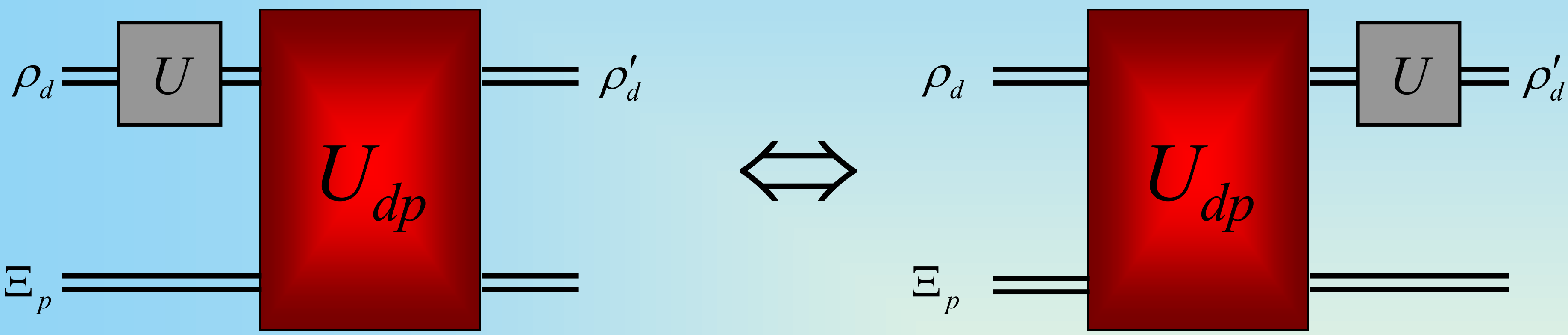
$$P_\theta = \cos\theta \mathbf{1} + \sin\theta S \quad S(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle, \ \dim \mathcal{H}_p = \dim \mathcal{H}_d$$

program Ξ_p encodes contractive operation Φ_Ξ with the only fixed point Ξ_d

$$\Phi_\Xi[\Xi_d] = \Xi_d$$

Covariant Processors

- the covariance condition



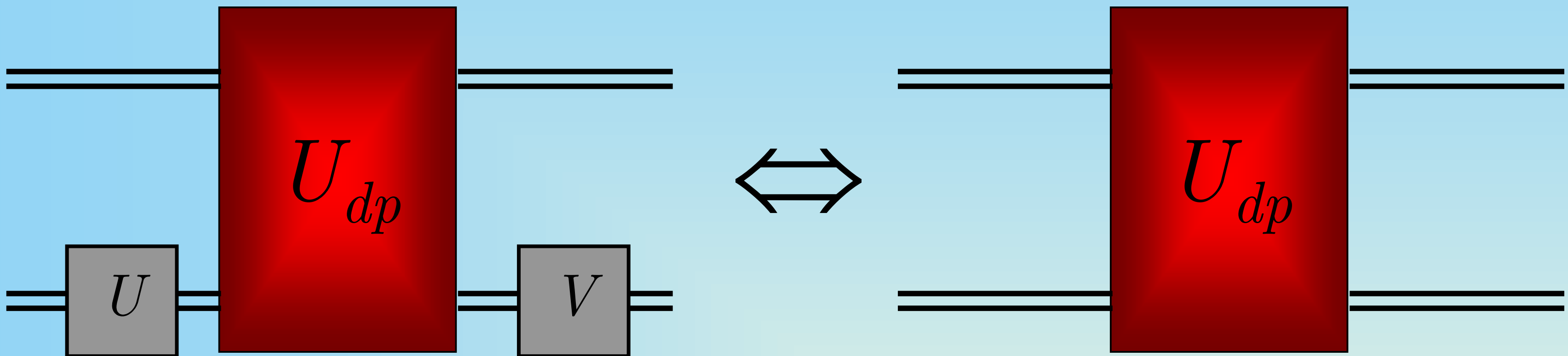
$$\Phi_{\Xi}\left[U\rho_d U^{\dagger}\right]=U\Phi_{\Xi}\left[\rho_d\right]U^{\dagger}$$

$$\forall \Xi \in S \subset S\left(H_p\right)$$

$$\forall U \in SU\left(\dim H_d\right)$$

- covariance with respect to the program states from the set S
- if $S = S(H)$, then the existence of nontrivial covariant U_{dp} is questionable
- if U_{dp} implements unitary transformation $\Rightarrow$ it is not covariant in this sence

Equivalent Processors



$$\text{if} \quad U_{dp}^{(2)} = (\mathbf{1} \otimes V)\, U_{dp}^{(1)}\, (\mathbf{1} \otimes U), \quad \text{then} \quad \boxed{U_{dp}^{(1)} \sim U_{dp}^{(2)}}$$

- equivalent processors implement the same quantum operations
- basis operators $A_{jk}^{(1)}$ and $A_{jk}^{(2)}$ associated with the two processors satisfy

$$A_{jk}^{(2)} = \sum_{m,n=1}^{N} (U_{p1})_{jm} (U_{p2})_{nk} A_{mn}^{(1)}$$

Inverse Problem

"Given a set Φ_x of quantum operations . Is it possible to design a processor that performs all these operations?"

1. Continous set of unitaries = question of universality NO
2. Phase damping channel = model of decoherence YES
3. Amplitude damping channel = model exponetial decay NO

Conclusions & Open Questions

- programmable quantum computer – programs via quantum states programs can be outputs of another QC
- some CP maps via unconditional quantum processors
- arbitrary CP maps via probabilistic programming
- controlled information distribution (eavesdropping)
- simulation of quantum dynamics of open systems
- set of maps induced by a given processor (loops)
- quantum processor for a given set of maps
- quantum multi-meters

M.Hillery, V.Buzek, and M.Ziman: *Phys. Rev. A* 65, 022301 (2002).

M.Dusek and V.Buzek: *Phys. Rev. A* 66, 022112 (2002).

M.Hillery, M.Ziman, and V.Buzek: *Phys. Rev. A* 66, 042302 (2002)