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Quantum Error-Correcting Codes

Markus Grassl

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Arbeitsgruppe *Quantum Computing*
Prof. Dr. Thomas Beth
Institut für Algorithmen und Kognitive Systeme
Universität Karlsruhe, Germany

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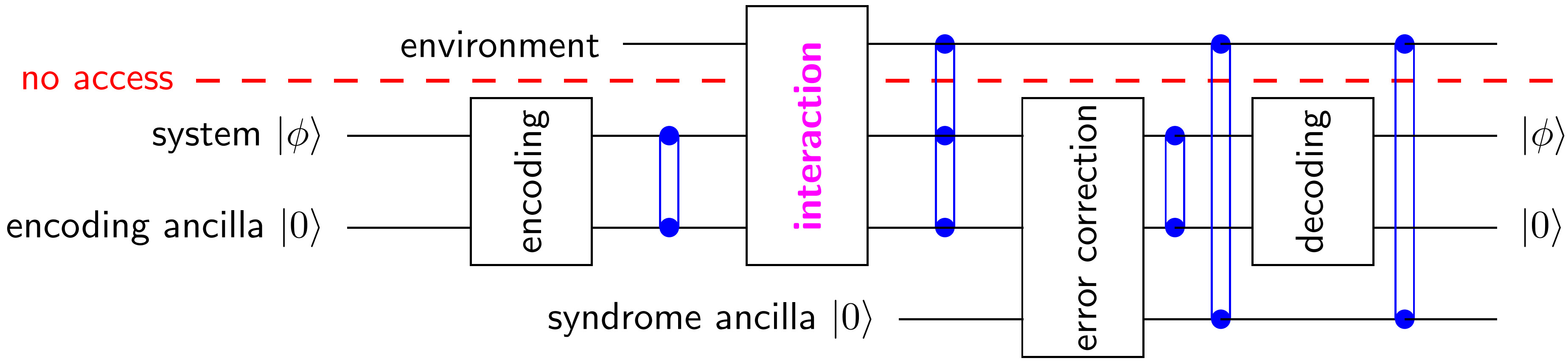
Quantum Error-Correction

General scheme



Basic requirement

knowledge about the interaction between the system and the environment

Common assumptions

- no initial entanglement between system and environment
- local or uncorrelated errors, i. e., only a few qubits are disturbed
  $\Longrightarrow$ CSS codes, stabilizer codes
- interaction with symmetry
  $\Longrightarrow$ decoherence free subspaces/subsystems

Quantum Error-Correction Codes

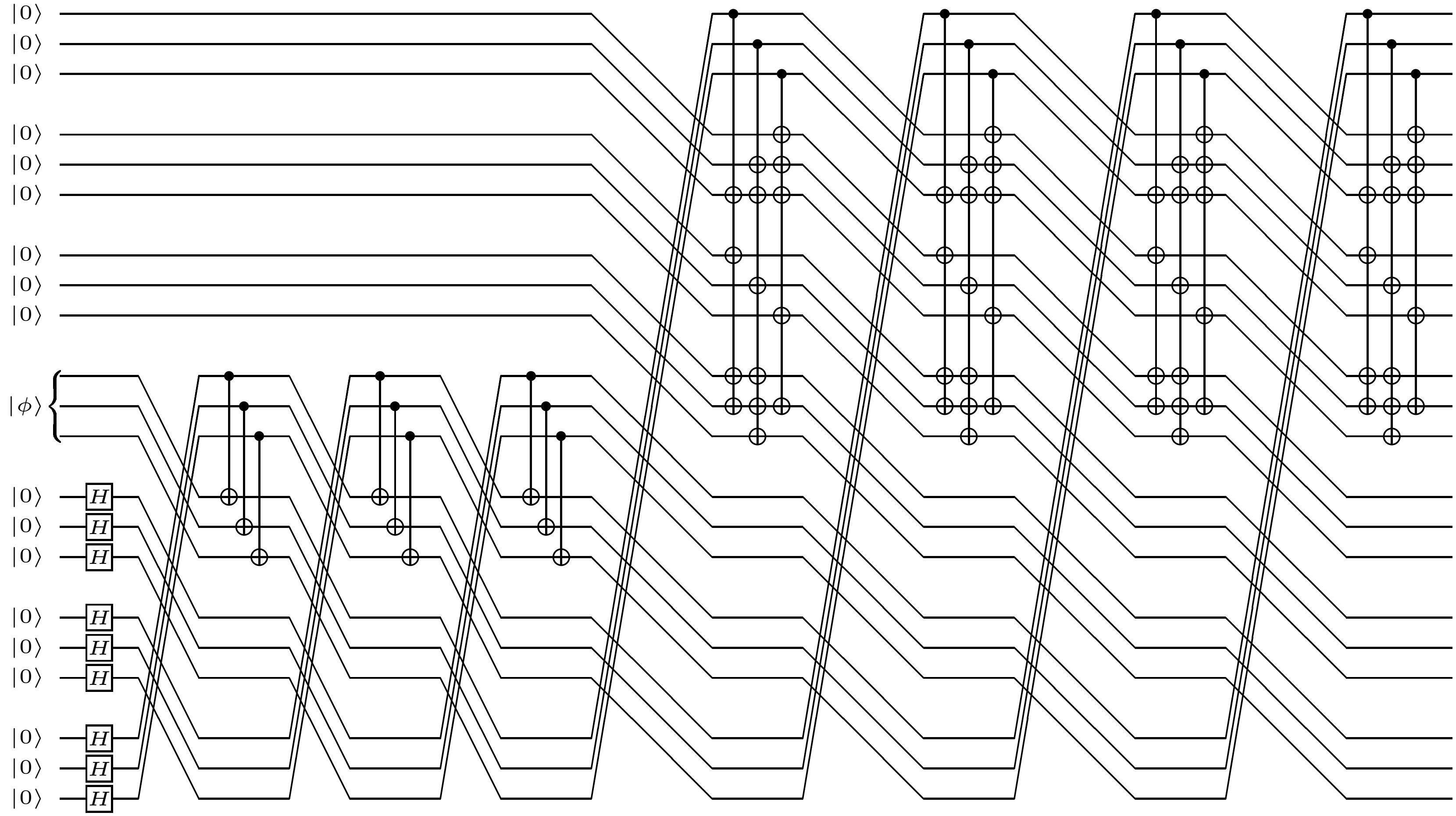
Constructions

- CSS codes, stabilizer codes [Calderbank, Gottesman, Rains, Shor, Sloane, Steane] based on classical error-correcting codes
- non-additive codes [Rains et al. 97] a non-additive code $C = ((5, 6, 2))$ exists, but no stabilizer code
- Clifford codes [Knill 96, Klappenecker & Rötteler 01] generalizing stabilizer codes

Algorithms

- quantum circuits for encoding & syndrome computation "easy" for CSS codes, for additive codes [Cleve & Gottesmann 97, Grassl 01]
- various algorithms for cyclic codes [Grassl et al. 99, Grassl & Beth 99]
- encoding based on interaction graphs [Schlingemann & Werner 01]

Encoder Based on Quantum Shift-registers



Encoder for the quantum Reed-Solomon code $[[21, 3, 5]]$ using quantum shift registers for the multiplication by $\tilde{\boldsymbol{g}}(X) = X + 1$ and $\boldsymbol{g}^{\perp} = \alpha X^3 + X^2 + \alpha^2 X + 1$.

Graph Codes

The ingredients:

- alphabet $A = \mathbb{F}_p^m$ of size $\alpha := |A| = p^m$
- weighted undirected graph Γ on $k+n$ nodes
- symmetric bicharacter χ on $A \times A$

Definition: A graph code is spanned by the vectors

$$|\underline{x}\rangle = \frac{1}{\sqrt{\alpha^n}} \sum_{y \in N} \Big(\prod_{\substack{i,j=1 \\ i<j}}^{k+n} \chi(z_i, z_j)^{\Gamma_{ij}} \Big) |y\rangle,$$

where $x \in A^k$ and $z = x + y \in A^k \times A^n$.

for qubits: $\displaystyle\prod_{\substack{i,j=1 \\ i<j}}^{k+n} \chi(z_i, z_j)^{\Gamma_{ij}}$ corresponds to the phase due to couplings $\sigma_z^{(i)} \sigma_z^{(j)}$

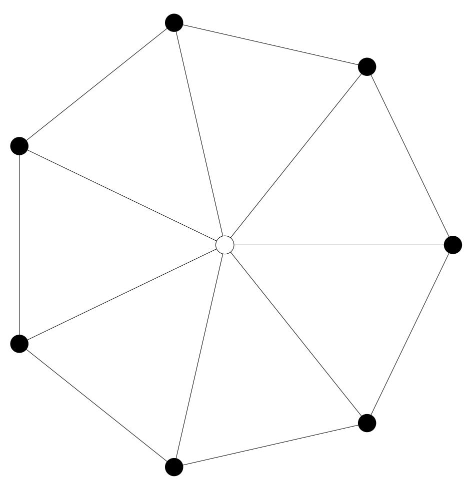
Graph Codes and Stabilizer Codes

[Schlingemann & Werner; Grassl, Klappenecker & Rötteler]

"$\Longrightarrow$" Each graph code is a stabilizer code.

Example:

The graph code corresponding to the wheel W_7



$$\Gamma_{W_7} = \left(\begin{array}{c|ccccccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

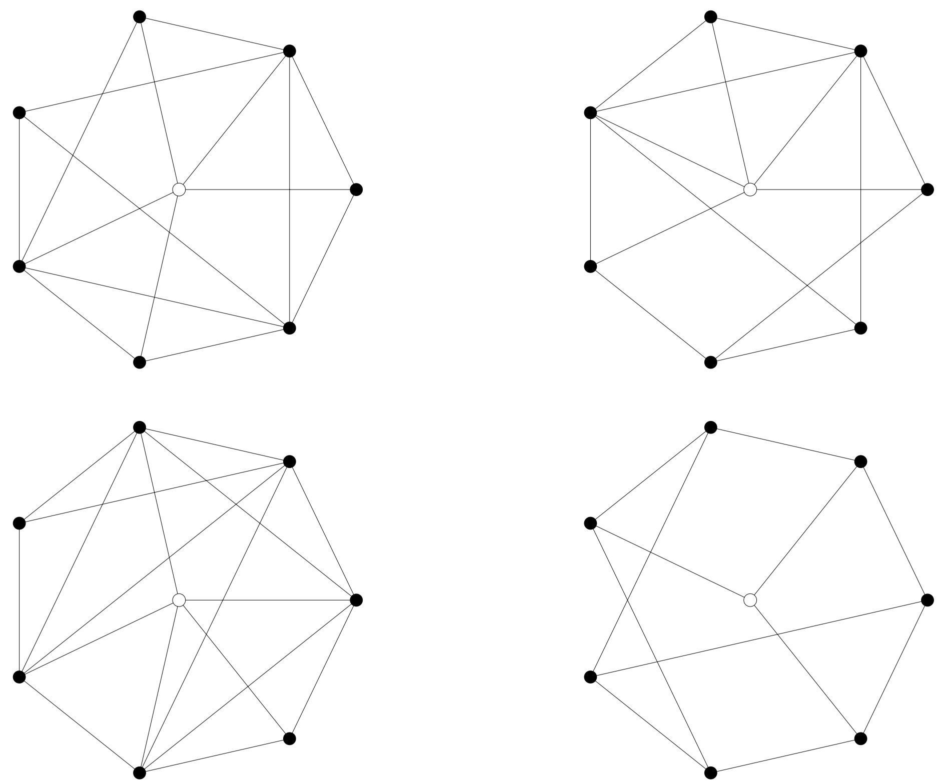
is a $[[7, 1, 3]]$ stabilizer code (which is not $GF(4)$-linear).

Graph Codes and Stabilizer Codes (contd.)

"$\Longleftarrow$" Each stabilizer code over $\mathbb{F}_q$ corresponds to a graph code (but the graph is not unique).

Example:

The CSS code $[\![7,1,3]\!]$ yields to non-isomorphic graphs

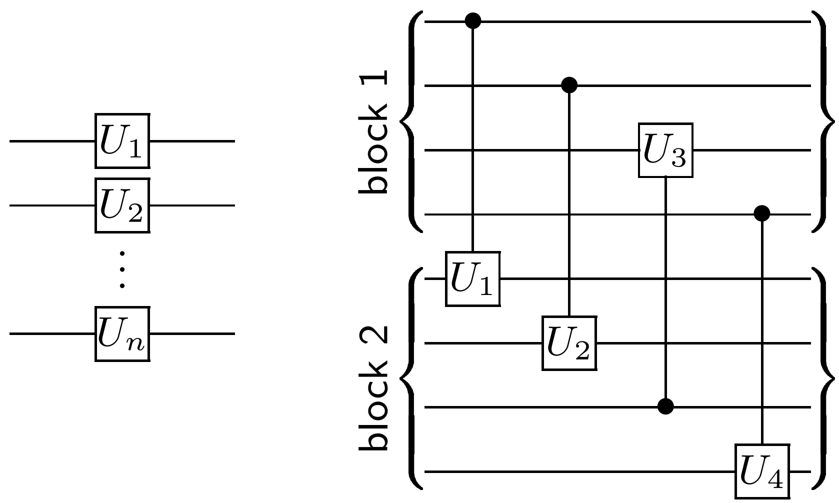


$\Longrightarrow$ alternative interaction graphs for the encoding

Fault Tolerant Quantum Computing

see e. g. [Aharonov & Ben-Or, Knill & Laflamme, Preskill, Steane]

- **encoded operations**: map codewords to codewords
- prevent spreading of errors



local operations   transversal operations

- fault tolerant operations also for error correction
  $\Longrightarrow$ requires supply of "fresh qubits" and
  fault tolerant preparation/testing of states

Concatenated Codes

Knill et al., *Resilient quantum computation* 377

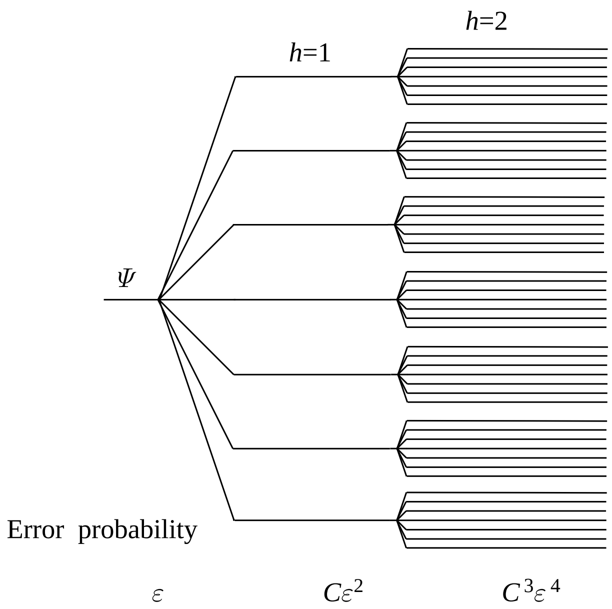


Figure 7. Concatenation of the seven-bit code. If the error rate is ϵ for the qubits, the encoding will gives a rate of $C^{2^h-1}\epsilon^{2^h}$ for the hth level of the hierarchy.

- many levels of error correction
  $\Longrightarrow$ reduction of the error probability
- (parallel) operations in each level
  $\Longrightarrow$ new errors due to imperfect gates

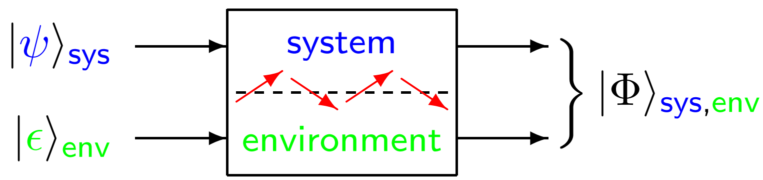
$\Bigg\}$ threshold τ for the error probability & gate errors
$\tau \approx 10^{-4}$–10^{-3} [Steane 02]

Decoherence Free Subspaces/Subsystems (DFS)

see e. g. [Zanardi & Rasetti 97; Lidar; Knill] and many more

also called: noiseless subspaces/subsystems, passive error-correction, error-avoiding codes

Main idea: *"Correct errors before they occur"*



known interaction (Hamiltonian)

decomposition of the interaction algebra $\mathcal{A}$ and the Hilbert space $\mathcal{H}$

$$\mathcal{A} \cong \bigoplus_j \mathbb{1}_{n_j} \otimes M(d_j, \mathbb{C}) \qquad \mathcal{H} \cong \bigoplus_j \mathbb{C}^{n_j} \otimes \mathbb{C}^{d_j}$$

irreducible components of dimension d_j and multiplicity n_j

$\Longrightarrow$ for $d_j = 1$ exists an decoherence free subspace of dimension n_j
(for $d_j > 1$ decoherence free subsystem)

Problem: requires non-trivial symmetry of the interaction

DFS: Fault Tolerant Operations

operations in the algebra

$$\mathcal{A}' \cong \bigoplus_j M(n_j, \mathbb{C}) \otimes \mathbb{1}_{d_j}$$

commute with the interaction algebra

$$\mathcal{A} \cong \bigoplus_j \mathbb{1}_{n_j} \otimes M(d_j, \mathbb{C})$$

$\Longrightarrow$ those operations preserve the DFS

For some models, universal computation is possible based on the exchange Hamiltonian or other two-qubit interactions (see e. g. [Kempe et al. 00, DiVincenzo et al. 00]).

but: entangling gates require in general an embedding

$$\mathsf{DFS} \otimes \mathsf{DFS} \subset \widetilde{\mathsf{DFS}}$$

$\Longrightarrow$ larger $\widetilde{\mathsf{DFS}}$ based on even more symmetry

DFS: Further Aspects

Collective Decoherence

the interaction algebra is invariant under particle permutations

"the bath cannot distinguish between the particles"

$\Longrightarrow$ highly symmetric interaction

Problem: in general, lack of symmetry yields multiplicity $n_j = 1$

$\Longrightarrow$ simulation of an effective interaction Hamiltonian:

apply (fast) local operations

Problem: not robust against gate errors

(e. g. the exchange interactions must be able to address individual particles)

$\Longrightarrow$ combination with active QECC

- using DFS as single "qudits" for a QECC (e. g. [Lidar et al. 98])
- embedding an active QECC into a DFS (e. g. [Plenio et al. 97, Alber et al. 01])

Jump Codes

(cf. Alber et al., PRL vol. 86, no. 19, pp. 4402–4405, May 7, 2001, quant-ph/0103042)

Quantum jump

$$\begin{tikzcd} |1\rangle \arrow[r] \arrow[dr, "\text{emission of e.g. photons}"] & |1\rangle \\ |0\rangle \arrow[r] & |0\rangle \end{tikzcd}$$

Effective Hamiltonian (no jump, but monitoring)

$$H_{\mathsf{eff}} = \sum_{\nu=1}^{n} -i\hbar\,\Gamma\,|1\rangle_\nu\langle 1|_\nu \qquad U_{\mathsf{eff}}(t) = \prod_{\nu=1}^{n} \exp\left(-t\,\Gamma\,|1\rangle_\nu\langle 1|_\nu\right)$$

$\Longrightarrow$ decoherence free subspace (DFS): constant number of excited states $|1\rangle$

additionally: correct errors due to *detected quantum jumps*, i. e.,
errors at known positions (classical side information)
$\Longrightarrow$ "quantum erasure channel"

QECC: Possible Directions to Proceed

Higher dimensional subsystems

- individual quantum systems are not only two-dimensional
- generalization of stabilizer codes [Rains 99, Ashikhmin & Knill 2001]
- for large alphabets, quantum MDS codes exist [Rains 99, Schlingemann & Werner 01]

Refined error models

- find systems where local/collective errors are dominant
- use additional side information [Grassl et al. 96, Gregoratti & Werner 02]
- impose symmetries [Zanardi 98, Viola et al. 00]

Optimize both QECC & algorithms

- (near) optimal codes for small systems
- better methods of fault tolerant error correction (e. g. [Steane 02])
- robust algorithms (e. g. approximative Fourier transform)

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