Accessibility of physical states and non-uniqueness of entanglement measure

Fumiaki Morikoshi,[1,2,*] Marcelo França Santos,[2,†] and Vlatko Vedral[2,‡]

[1]*NTT Basic Research Laboratories, NTT Corporation*
3-1 Morinosato-Wakamiya, Atsugi-shi, Kanagawa, 243-0198, Japan
[2]*Optics Section, The Blackett Laboratory, Imperial College,*
Prince Consort Road, London, SW7 2BW, United Kingdom

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Ordering physical states is the key to establishing a unique measure of some physical quantity related to those states. To achieve ordering, it is important to check accessibility between two physical states by some physical process. When there exists an *operation* that converts one state to another, these states can be ordered in respect to this particular process. This ordering (together with a few other natural assumptions) makes it possible to define a quantity that compares the states. However, if it is impossible to convert one state into another in either direction within a given framework, there exists no coherent way to compare those two states. When all elements in a given set of physical states can be completely ordered, i.e., arbitrary two states can be ordered (total order), we can make at least one consistent measure that quantifies the set. However, if there exists no ordering that works globally, i.e., a certain pair of states cannot be ordered (partial order), then we fail to find a consistent way to “align” all the states.

One of the most familiar examples in physics that contain incomparable states is in special theory of relativity. A pair of events in the space-time that include each other in their light-cone (i.e., time-like interval) is comparable because one can affect the other by sending some signal. However, if one is outside of the light-cone of the other (i.e., space-like interval), then it is impossible to connect them by any physical operation. Therefore, there exists no unique way of ordering such two states (for example, see Chapter 17 of [1]).

On the other hand, the most beautiful and successful application of the theory of ordering physical states is in thermodynamics, where all equilibrium states are ordered using a unique measure of entropy. Given two equilibrium states (A and B), entropy S distinguishes possible and impossible directions of adiabatic processes between the two states; If $S(A) \leq S(B)$ then A can access B via an adiabatic process. (If the equality holds, B can also access A and so the process becomes reversible.)

In order to clarify the structure of thermodynamics and prove the uniqueness of entropy, Giles developed a rigorous set of mathematical axioms [2]. The crux of Giles's approach to thermodynamics is that the existence of the unique measure of entropy depends heavily on accessibility between different physical states. Axiom 5 in his formalism, which is in some sense the most nontrivial one of the axioms, is phrased as follows: *If two states A and B are both accessible from another state C, then A and B are accessible each other in either direction (or both).* (For other natural axioms and further details of Giles's approach, see Refs. [2, 3].)

It has been shown recently that thermodynamics and theory of quantum entanglement share the same mathematical structure. Adiabatic processes in thermodynamics correspond to manipulations of bipartite entangled pure states by local operations and classical communication (LOCC) in the context of quantum information theory [3]. Thus, entropy gives a unique measure in this context as well (known as the von Neumann entropy of entanglement [4, 5]).

Quantum entanglement has been a subject of intensive research because it is a new resource in physics as well as an indispensable resource in quantum information processing. As in the case of other physical resources, it is desirable to find a unique measure of entanglement in order to exploit it effectively and efficiently. Contrary to the case of bipartite pure states, a unique measure of entanglement in mixed states has not been established yet.

In this work, we prove that, if one is restricted to LOCC, there exists no unique measure of entanglement in bipartite mixed states by invoking Giles's axioms. We show the non-uniqueness of entanglement measure by giving a counterexample to Axiom 5.

In particular, we show that once we go into the mixed-state regime, interconvertibility between arbitrary two states vanishes even in the asymptotic setting. The rigorous definition of interconvertibility here is as follows: A state ρ is convertible into a state σ if and only if for every (arbitrarily small) real number ϵ, there exists an integer n_0, and a sequence of LOCC L_n such that for any integer $n \geq n_0$ we have that

$$\parallel L_n(\rho^{\otimes n}) - \sigma^{\otimes n} \parallel \leq \epsilon, \tag{1}$$

where $\rho^{\otimes n} = \rho \otimes \rho \cdots \otimes \rho$ represents a tensor product of n copies of the state ρ and $|| \cdots ||$ denotes the usual trace norm distance between two mixed quantum states. Loosely speaking, one state can be converted into another if a certain number of copies of the former can arrive at an arbitrarily good approximation of the *same* number of copies of the latter via LOCC in the asymptotic limit.

Intuitively, bipartite mixed states that are most likely to fail this axiom are bound entangled states [6]. Since bound entangled states are mixed states from which no entangled pure state can be distilled, if we take one of those and a pure entangled state as a pair of possible candidates for a counterexample, the first half of the proof has already been accomplished by definition. So, all we have to do is to prove the inconvertibility in the opposite direction.

In order to prove that, we take a particular bound entangled state constructed from an unextendible product basis (UPB) [7]. Suppose Alice and Bob have three-level quantum systems (qutrits), respectively. Consider the following five orthogonal product bases:

$$\begin{array}{rclcl} |\psi_1\rangle &=& |0\rangle &\otimes& (|0\rangle+|1\rangle) \\ |\psi_2\rangle &=& (|0\rangle+|1\rangle) &\otimes& |2\rangle \\ |\psi_1\rangle &=& |2\rangle &\otimes& (|1\rangle+|2\rangle) \\ |\psi_4\rangle &=& (|1\rangle+|2\rangle) &\otimes& |0\rangle \\ |\psi_5\rangle &=& (|0\rangle-|1\rangle+|2\rangle) &\otimes& (|0\rangle-|1\rangle+|2\rangle). \end{array} \tag{2}$$

These bases form a UPB, which means that there exists no product state orthogonal to all of the above five bases. Consequently, the four-dimensional subspace complementary to this five-dimensional one does not contain any product states. Therefore, the projection operator onto this complementary space

$$\rho_{\mathrm{AB}} = \frac{1}{4}\left(\mathbb{1} - \sum_{i=1}^{5} |\psi_i\rangle\langle\psi_i|\right) \tag{3}$$

turns out to be an entangled state. The important fact about the state ρ_{AB} is that its entanglement cost $E_C(\rho_{\mathrm{AB}})$ is positive [8], which is defined as $E_C(\rho) \equiv \lim_{n\to\infty} E_f(\rho^{\otimes n})/n$ [9], where $E_f(\rho)$ represents the entanglement of formation of ρ [10]. Owing to this property, one can choose an entangled pure state $\sigma_{\mathrm{AB}} = |\phi\rangle\langle\phi|$ such that

$$0 < E_C(\sigma_{\mathrm{AB}}) < E_C(\rho_{\mathrm{AB}}). \tag{4}$$

For simplicity, we choose $|\phi\rangle$ to be an entangled states with Schmidt number two or three, i.e., a 2×2 or 3×3 system. Since the entanglement cost E_C is an entanglement monotone, i.e., it cannot increase under LOCC, n copies of σ_{AB} can never be converted into the same number of copies of ρ_{AB} even asymptotically. At the same time note that a maximally entangled state $|\Phi_3\rangle_{\mathrm{AB}}$ can access both ρ_{AB} and σ_{AB} without reducing the number of copies. Therefore, we found a counterexample that two states ρ_{AB} and σ_{AB} are *not* interconvertible into each other in spite of the fact that both of them can be accessed from the same state $|\Phi_3\rangle_{\mathrm{AB}}$. According to Giles's argument, this immediately results in the nonexistence of the unique measure of entanglement under LOCC.

We have therefore proven that there is no way to construct the unique measure of entanglement for mixed states at least under ordinary LOCC. Since there is no operational way to link incomparable states, there seems to be no way of assigning "meaningful" amounts of entanglement to them that could determine which state is more entangled.

The non-uniqueness of entanglement measure in mixed states could be implicitly expected from the fact that distillation and formation processes are generally irreversible even in the asymptotic limit. However, we gave a rigorous proof of the non-uniqueness here from a more general and formal point of view that might be applicable to a broader range of theoretical physics.

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* Electronic address: `fumiaki@will.brl.ntt.co.jp`
† Electronic address: `m.santos@ic.ac.uk`
‡ Electronic address: `v.vedral@ic.ac.uk`

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