Research Center for Quantum Information

Entangled Graphs

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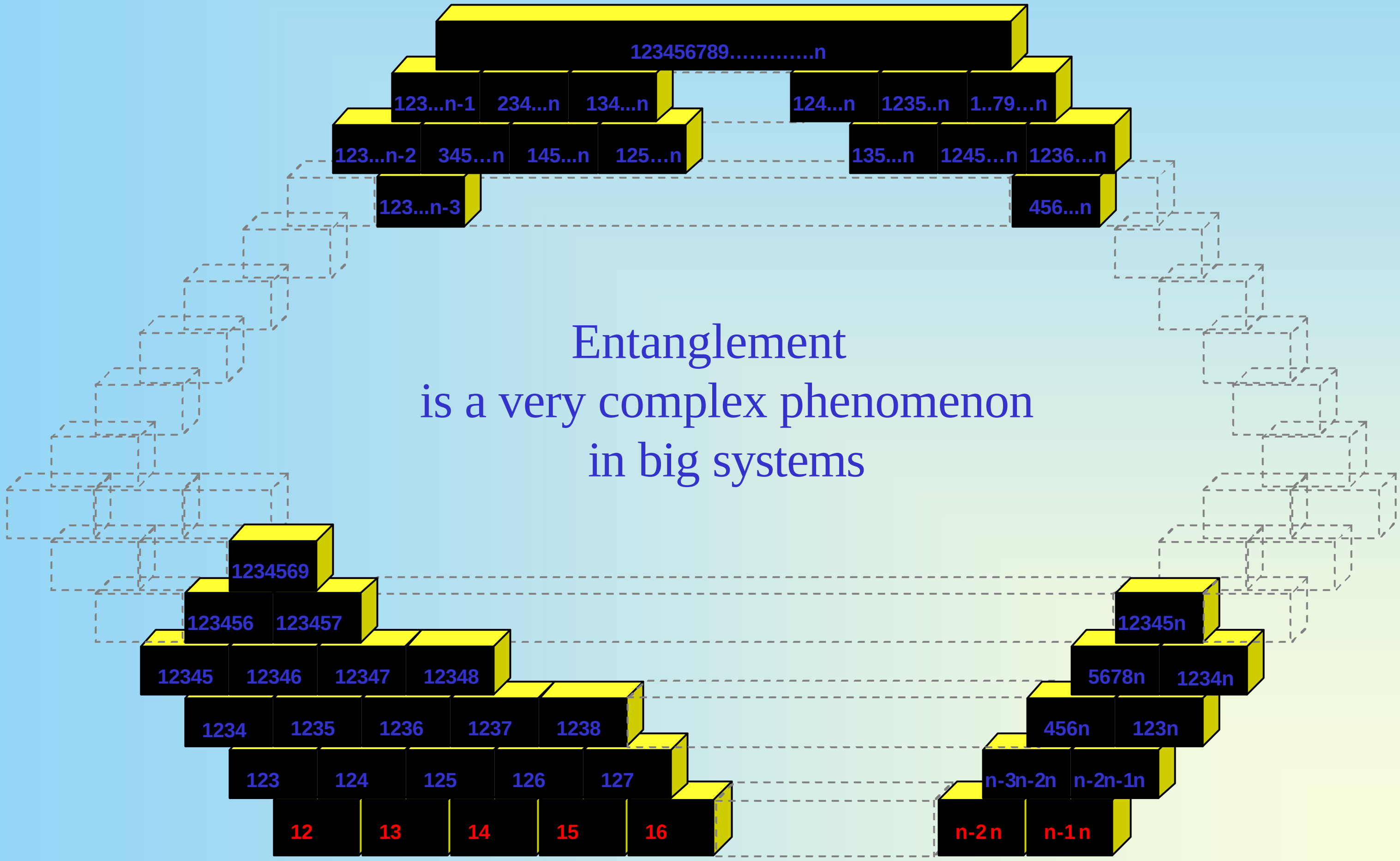
Entanglement



Entanglement
is a very complex phenomenon
in big systems

- Interesting feature: Limited sharing [CKW inequalities]

V. Coffman, J. Kundu, W. Wootters: *Phys.Rev. A***61** (2000) 052306

Classical Correlations

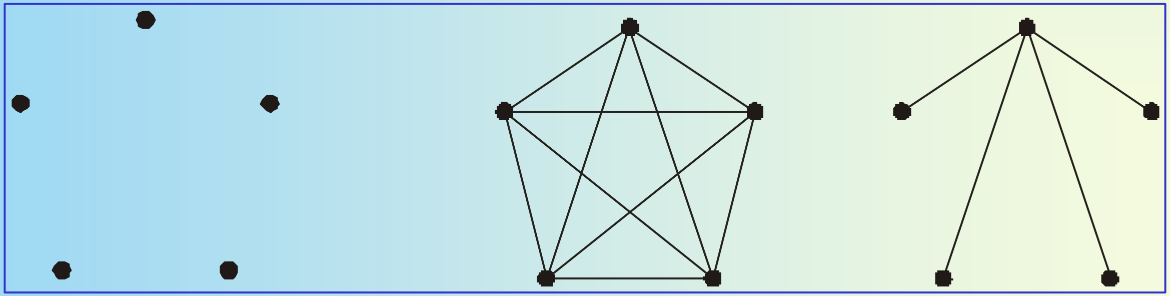
- Correlation in quantum systems has two principal origins:
  - Correlation induced by entanglement
  - Correlation due to *statistical mixing*
- More-partite entanglement fragments into bipartite correlation
- Problem with a suitable measure, that could be compared to concurrence
- Basic question: which bipartite entanglement and classical correlation configurations are allowed?

Forerunners

- Entangled chains – long chain of entangled qubits
  W. Wootters, `quant-ph/0001114` (2000)
- Entangled webs – N qubits pairwise entangled
  M. Koashi, V. Buzek, N. Imoto *Phys. Rev. A* **62**, *050302(R)-1–4 (2000)*.
- Entangled molecules – entanglement engineering on mixed states
  W. Dur, *Phys. Rev. A* **63**, *020303(R) (2001)*.
- No conditions on separability
- Classical correlation were not considered at all

Entangled Graphs

- Particle (qubit) = vertex
- Entanglement between 2 particles = edge
- NO edge implies NO entanglement
- The graph is defined by the number of qubits *N* and a set of edges *S*
- $\{i, j\} \in S \Leftrightarrow$ particle i and j are entangled
- $k = |S|$ is the number of entangled pairs in the system



Mixed States

- For any specified graph, we search for a mixed state, which would be characterized by that graph
- Take a state

$$|\psi\rangle_{ij} = |\psi^+\rangle_{ij}\,|0\ldots0\rangle_{rest}$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

and make a mixture

$$\rho = \frac{1}{k}\sum_{\{i,j\}\in S} |\psi\rangle_{ij}\langle\psi|_{ij}$$

- The concurrence is non-zero only for pairs $\{i, j\} \in S\ ; C_{ij} = \frac{1}{k}$

Pure States

- In general, the problem is more complicated
- Number of free parameters are limited in comparison to mixed state ($2^{n+1}-2$ in comparison to $2^{n^2}-1$)
- The same approach as for mixed states (combination of Bell states) does not work
- However, we are able to formulate a theorem:

> For each entangled graph there exists at least one pure state

Martin Plesch a Vladimír Bužek, *Entangled Graphs: Bipartite entanglement in multi-qubit systems*, Phys. Rev. A **67**, 012322 (2003)
Martin Plesch a Vladimír Bužek, *Entangled graphs*, Quantum Information and Communication 2, 530-539 (2002)

Constructive Proof

- The state vector can be shown to be:

$$|\Psi\rangle = a\,|0\ldots0\rangle + b\,|1\ldots1\rangle + \frac{g}{\sqrt{k}} \sum_{\{i,j\}\in S} |1\rangle_i\, |1\rangle_j\, |0\ldots0\rangle_{rest}$$

$$|g|^2 \le \frac{4|a|^2}{k};\ |g|^2 \le |ab|;\ |a|^2 + |b|^2 + |g|^2 = 1$$

- The concurrence is non-zero only for wished pairs and is determined by parameters of the state
- We use only a small, N^2-dimensional part of the whole Hilbert space in comparison to the total dimension 2^N

Weighted Graphs for Pure States

- Edges in graphs are weighted by concurrence
- Definitely not all graphs have representatives (CKW inequalities and)
- If we post a strict condition for maximal concurrence $C_{\max} \sim 0.24 / N$, we can show that

For each weighted entangled graph, where $C_{ij} \leq C_{\max}$, there exists a pure state

M. Plesch, J. Novotný, Z. Dzuráková, V. Bužek, *Controlling of bipartite entanglement in many-partite states*, sent to IOP

Weighted Graphs for Pure States

- The state vector is:

$$|\Psi\rangle = a\left(|0\ldots0\rangle + |1\ldots1\rangle\right)$$

$$+\sum_{\{i,j\}} g_{ij}\left(|1\rangle_i|1\rangle_j|0\ldots0\rangle_{rest} + |1\rangle_j|1\rangle_i|0\ldots0\rangle_{rest}\right)$$

$$2\,|a|^2 + 2\sum_{\{i,j\}}\left|g_{ij}\right|^2 = 1$$

- Concurrence:

$$C_{ij} = 2\left(a g_{ij} - \sum_{\{i,k\}}{}' |g_{ik}{}^2| - \sum_{\{j,k\}}{}' |g_{jk}{}^2|\right)$$

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Weighted Graphs for Pure States

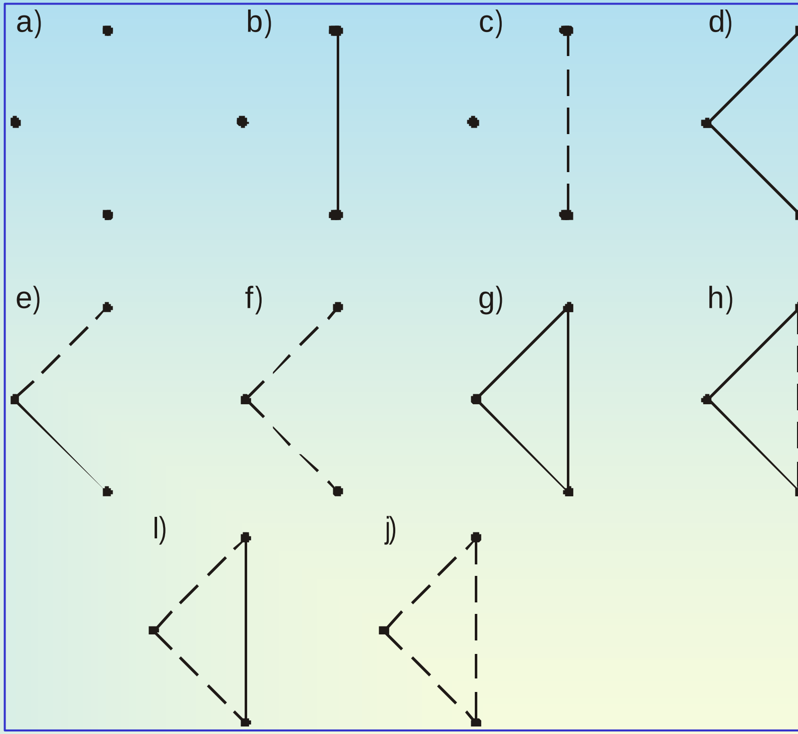
- There exists a procedure to find g_{ij} for given C_{ij}
- We start with $g_{ij} \equiv g_{\max} \rightarrow C_{ij} \equiv C_{\max}$
- Step by step we lower gammas to update the concurrence
- In every step, every concurrence is greater than or equal to the desired concurrence (we approach the desired state *from to top* in the viewpoint of concurrence)
- After every step, gammas are smaller than before – the sequence is convergent
- Again, we use only the N^2-dimensional part of the Hilbert space

Classical Correlations

- Given a state of N qubits, a pair of them is correlated, iff its density matrix is correlated
- A density matrix uncorrelation condition: $\rho_{ij} = \rho_i \otimes \rho_j$
- There are three types of states of two qubits:
  - Entangled pair – full line
  - Correlated, but not entangled pair – dashed line
  - Not correlated, factorized pair – no line

Graphs with Classical Correlations

- The graph is given by
  - The set of entangled pairs S^E
  - The set of correlated pairs S^C
- For the definition of the state vector one needs to specify
  - The number of qubits correlated with the i-th qubit m_i
  - The total number of correlated pairs M



Mixed States

- We can utilize the true classical correlation coming out of the classical uncertainty of the state
- The state space is big
- On the other side, uncorrelation condition is, in comparison to entanglement, very tight
- Still, we are able to proof, that

> For each correlation graph there exists at least one mixed state

Martin Plesch a Vladimír Bužek, *Entangled graphs II: Classical correlations in multi-qubit entangled systems*, quant-ph/0306008, accepted in Phys. Rev. A

Constructive Proof

- The state vector is

$$r = \frac{1}{2(N-1)^2}\Big\{\big(N^2 - 3N + \tfrac{1}{2}M + 2\big)\big|0\ldots0\big\rangle\big\langle 0\ldots0\big| +$$

$$+\sum_{i=1}^{N}\big[(N-1) - \tfrac{1}{2}m_i\big]\big|0\ldots01_i0\ldots0\big\rangle\big\langle 0\ldots01_i0\ldots0\big| +$$

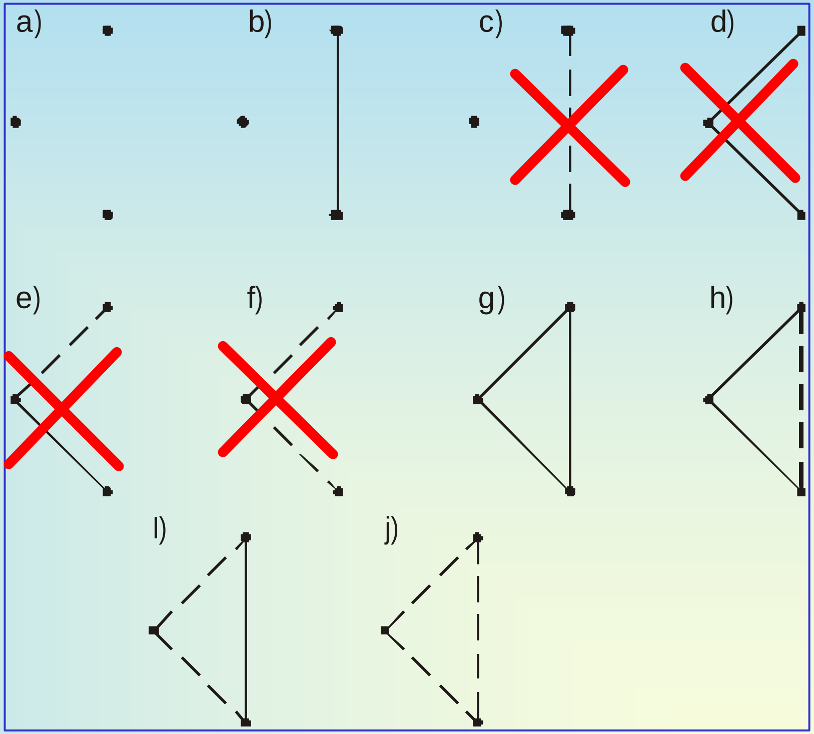
$$+\sum_{\{i,j\}\in S^E}\big(\big|0\ldots1_i\ldots0\big\rangle\big\langle 0\ldots1_j\ldots0\big| + \big|0\ldots1_j\ldots0\big\rangle\big\langle 0\ldots1_i\ldots0\big|\big) +$$

$$+\sum_{\{i,j\}\notin S^C}\tfrac{1}{2}\big(\big|0\ldots1_i\ldots1_j\ldots0\big\rangle\big\langle 0\ldots1_i\ldots1_j\ldots0\big|\big)\Big\}$$

- Moreover, density matrices of individual qubits are identical

Pure States

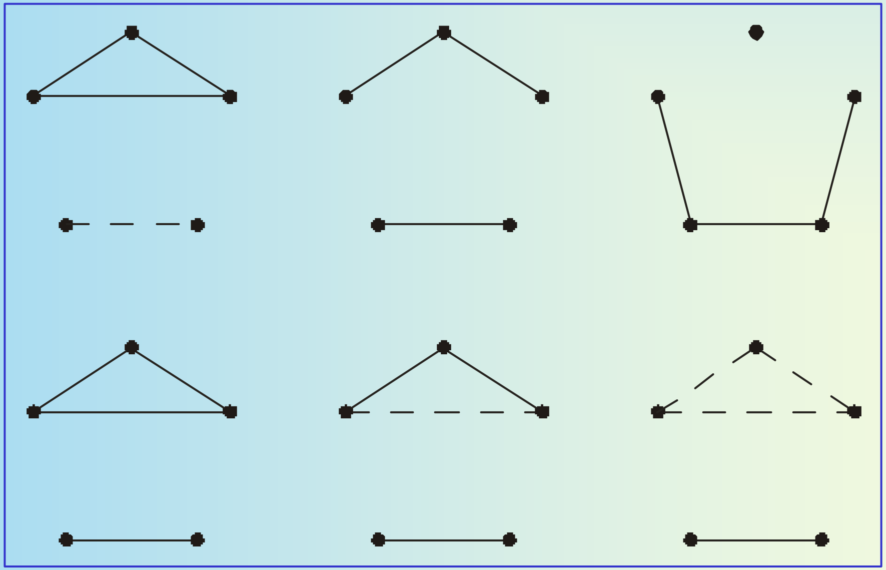
- We know that not all graphs with “double” constraints can be realized by pure states
- The not-realizable graphs have a common property – an open edge



Pure States

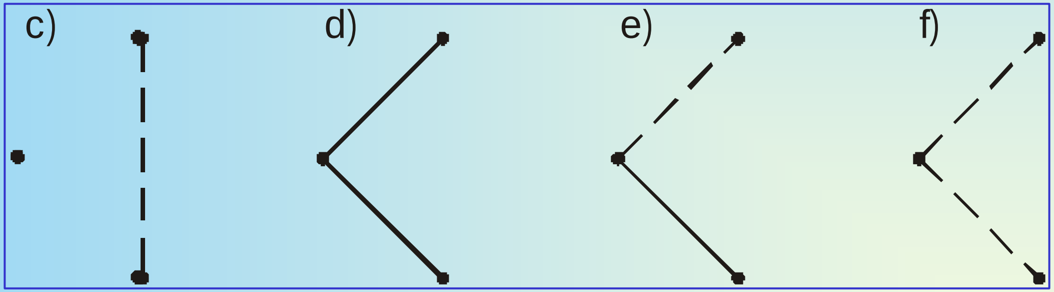
- In cases with more particles we are able to state only some general assertions

Pure states for unconnected correlation graphs exist, if they exist for fragments of the graph



Pure States

No pure states exist for correlation graphs with an open edge



For each correlation graph, where every pair of qubits is entangled or correlated, there exists at least one pure state

Conclusions

- We have proved a theorem of existence of all pure-state entangled graphs
- Theorem of existence of a class of pure-state weighted entangled graphs
- Theorem of existence of mixed-state correlation graphs
- Theorems of existence and non-existence of some classes of pure-state correlation graphs
- Problem numerically solved for 4 qubits (all ambiguous graph do exist)