Lectures on quantum computation by David Deutsch

Lecture 1: The qubit

Worked Examples

1. You toss a coin and observe whether it came up heads or tails.
   (a) Interpret this as a physics experiment that ends with a measurement. What is the measuring instrument?
   (b) Interpret this as a computation. What is the computer?

2. Give an example of a quantum observable in everyday life. Call your observable $\hat{X}$.
   (a) What is $\mathrm{Sp}\,\hat{X}$?
   (b) How could $\hat{X}$ be measured, in principle?
   (c) Pick a value in $\mathrm{Sp}\,\hat{X}$. How could $\hat{X}$ be prepared with that value, in principle?
   (d) Give at least two ways of measuring the observable $e^{\hat{X}}$.
   (e) Given an instrument that easily measures $e^{\hat{X}}$, how could $e^{2\hat{X}} - \hat{1}$ be easily measured?

3. Prove that if $\mathrm{Sp}(\hat{\theta})$ is the set of all real numbers between 10 and 170, $\mathrm{Floor}(\hat{\theta}/90)$ is a Boolean observable.

4. Given that the Boolean observable $\hat{X}$ has spectrum $\{a,b\}$, find a Boolean observable of the same system with spectrum {-1,1}.

5. Let $\hat{X}$ be an observable. Prove that there is no such observable as $e^{i\hat{X}}$ unless $\hat{X}$ satisfies a certain condition, and state that condition.

6. Show that the observable $\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ referred to in Lecture 1 does indeed have spectrum $\{-1,1\}$.

7. Show that the expectation value function that appears in Lecture 1, namely $\left\langle \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \right\rangle = a$, has all the properties required of such a function, namely
   (a) it is real-valued;
   (b) it is linear ($\langle \lambda\hat{X} + \mu\hat{Y} \rangle = \lambda\langle \hat{X} \rangle + \mu\langle \hat{Y} \rangle$); and

(c) Min Sp $\hat{X} \leq \langle \hat{X} \rangle \leq$ Max Sp $\hat{X}$.

8. Prove that in a quantum system S with the above expectation-value function, there exists an observable $\hat{X}$ such that every sharp observable of S has the form $\lambda \hat{X} + \mu \hat{1}$ for some real λ and μ.

9. Prove the *uncertainty principle for qubits,* in the form 'for any qubit, there exists no possible expectation value function such that every observable of that qubit is sharp'.

10. What would it feel like to be made of matrix-valued variables instead of real-valued ones? ('Made of…' is short for 'made of something whose properties are accurately represented by…'.)

Hints

1. (Coin toss)
   (a) In the example given in the lecture, the measuring instrument included the protractor and the light.
   (b) A computer is a physical object that is prepared with an input and later measured to obtain the output.

2. (Everyday observable).
   (a) The spectrum consists of real numbers.
   (b) Don't be deterred by expense, legality, morality, difficulty…
   (c) See 2(b) above.
   (d) Measure $\hat{X}$ first.
   (e) Measure $e^{\hat{X}}$ first.

3. Calculate the spectrum of $\mathrm{Floor}\left(\hat{\theta}/90\right)$.

4. What is the spectrum of $\lambda\hat{X} + \mu\hat{1}$?

5. What is the spectrum of $e^{i\hat{X}}$? Remember that observables are Hermitian.

6. The eigenvalues x of $\hat{X}$ are the solution of the characteristic equation $\det\left(\hat{X} - x\hat{1}\right) = 0$.

7. (Expectation value function that appears in Lecture 1.)
   (a) a must be real because … ?
   (b) Let $\hat{X} = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ and $\hat{Y} = \begin{pmatrix} d & e \\ e^* & f \end{pmatrix}$.
   (c) Work out the lower and higher eigenvalues $x_\pm$ of $\hat{X}$.

8. From the argument given in the lecture, a Boolean observable is sharp iff its expectation value equals one of its two eigenvalues.

9. Consider the family of Boolean observables $\hat{X}_\theta = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

10. *Would*?

Answers

1. (Coin toss.)
   (a) The light.
   (You could also include your own eye, parts of your brain … where you draw the line is up to you.)
   (b) The coin.
   (You could also include the hand that tosses the coin, the table it lands on … where you draw the line is up to you.) You provide the input to the computation by delivering a force to it in a particular way. You read the output by looking at (the light that comes from) the face that is uppermost when the coin stops moving.

2. Say, the closing price of a given share on the stock market on a given day.
   (a) The set of all real numbers of the form $k/100$ where k is a nonnegative integer (perhaps with some upper bound too).
   (b) By looking up the price in the following day's newspaper.
   (c) By causing the company to do well or badly, and then buying or selling suitable amounts of that stock during a period before the close of business.
   (d) Look up the value and raise e to the power of it. Or alter the rules of the stock market so that it announces stock prices on an exponential scale.
   (e) Measure $e^{\hat{X}}$. Then square the outcome and subtract 1.

3. When a real variable θ ranges between 10 and 170, $\text{Floor}(\theta/90)$ ranges over only two values, 0 and 1. Hence $\text{Floor}(\hat{\theta}/90)$ is a Boolean observable.

4. Let λ and μ be real numbers. The spectrum of $\lambda\hat{X}+\mu\hat{1}$ is $\{\lambda a+\mu, \lambda b+\mu\}$. Setting this equal to $\{-1,1\}$ and solving for λ and μ we find that the required observable is $\frac{2}{b-a}\hat{X}-\frac{a+b}{b-a}\hat{1}$. (Minus that observable would do just as well.)

5. The eigenvalues of an observable must be real. e^{ix} is real if and only if x is an integer multiple of π. So the condition is that $\text{Sp}\,\hat{X}$ contains only integer multiples of π.

6. Let x be in the spectrum of $\hat{X}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Then $\det(\hat{X}-x\hat{1})=\det\begin{pmatrix}-x&1\\1&-x\end{pmatrix}=x^2-1$. Hence $x=\pm1$.

7. (Expectation value function that appears in Lecture 1.)
   (a) a is a diagonal element of a Hermitian matrix and is therefore real;

(b) Let $\hat{X} = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ and $\hat{Y} = \begin{pmatrix} d & e \\ e^* & f \end{pmatrix}$. Then

$$\left\langle \lambda \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} + \mu \begin{pmatrix} d & e \\ e^* & f \end{pmatrix} \right\rangle = \lambda a + \mu d = \lambda \left\langle \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \right\rangle + \mu \left\langle \begin{pmatrix} d & e \\ e^* & f \end{pmatrix} \right\rangle.$$

(c) The eigenvalues $x_\pm$ of $\hat{X} = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ satisfy the characteristic equation $(a - x_\pm)(c - x_\pm) - |b|^2 = 0$. I.e. $x_\pm = \dfrac{a + c \pm \sqrt{4|b|^2 + (a-c)^2}}{2}$. Hence $x_+ \geq \dfrac{a+c}{2} + \left|\dfrac{a-c}{2}\right|$ and $x_- \leq \dfrac{a+c}{2} - \left|\dfrac{a-c}{2}\right|$. In other words, the higher of the two eigenvalues is greater than or equal to the higher of a and c, and the lower of the eigenvalues is less than or equal to the lower of a and c. But $\langle \hat{X} \rangle = a$, so in particular, $x_- \leq \langle \hat{X} \rangle \leq x_+$.

8. From the argument given in the lecture, a Boolean observable is sharp iff its expectation value equals one of its two eigenvalues. Let $\hat{X} = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, with the expectation value function $\langle \hat{X} \rangle = a$. This is sharp iff a is one of the eigenvalues $x_\pm$ of $\hat{X}$. But from the characteristic equation $(a - x_\pm)(c - x_\pm) - |b|^2 = 0$, a is one of those eigenvalues iff $b = 0$. Hence the sharp observables in this situation are precisely those whose matrix representations are diagonal. In other words those that are of form $\lambda \hat{X} + \mu \hat{1}$ where $\hat{X}$ is (say) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

9. Suppose that for some expectation value function, every observable of a particular qubit is sharp. Then in particular, all its observables of the form $\hat{X}_\theta = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ are sharp. From the characteristic equation $(\cos\theta + x)(\cos\theta - x) + \sin^2\theta = 0$, it follows that the eigenvalues of each $\hat{X}_\theta$ are ±1. But by the linearity of the expectation value function, $\left\langle \hat{X}_\theta \right\rangle = \cos\theta \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle + \sin\theta \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$. If we square both sides and substitute ±1 for all the expectation values, we obtain $\cos\theta \sin\theta = 0$, which for generic θ is a contradiction. Hence the premise that all the observables were sharp was false.

10. We *are* made of matrix-valued and not real-valued variables. It would feel exactly as we are feeling now.