## Lectures on quantum computation by David Deutsch

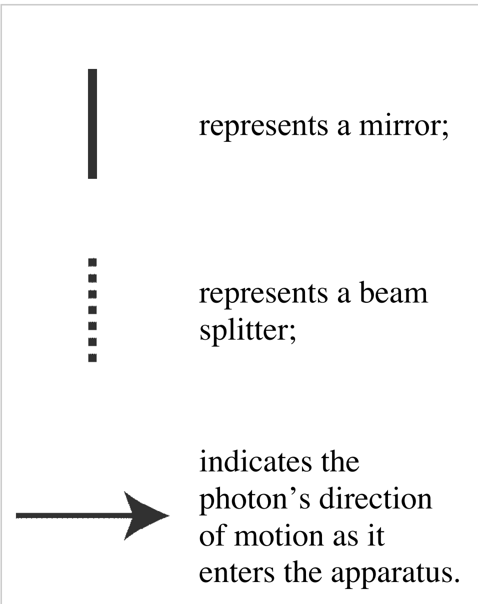
## Lecture 2: Interference

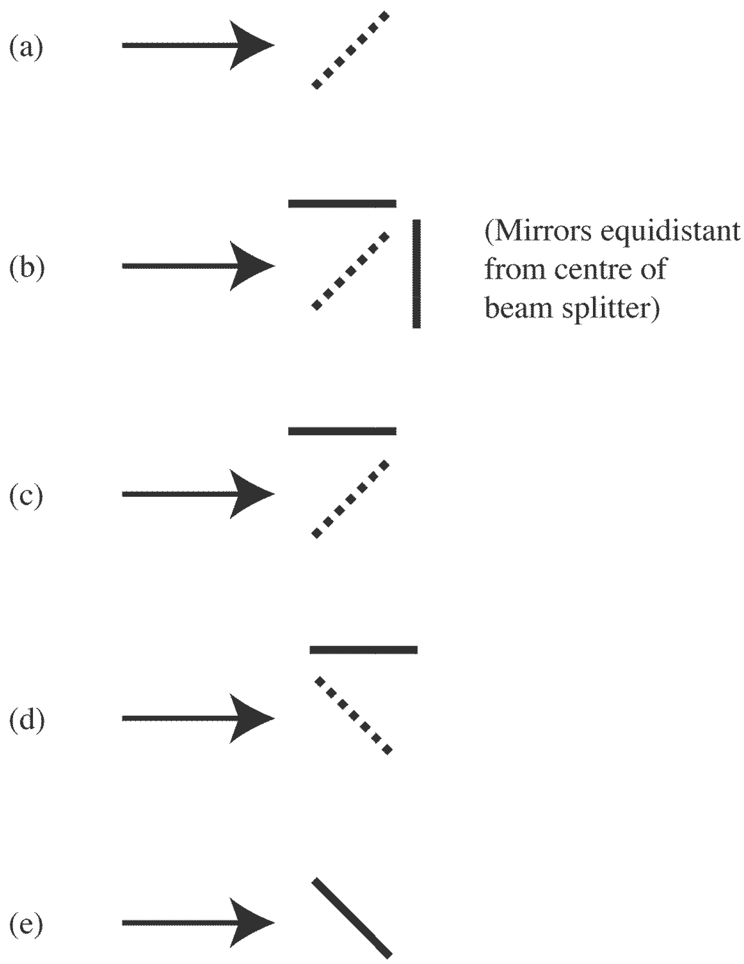
### Worked Examples

1. Suppose that photons are classical impenetrable spheres. Two photon beams each of cross-sectional area $1\text{mm}^2$ and each containing $3 \times 10^{15}$ photons per second pass through each other at right angles for one minute, and no photon is detected by an array of very sensitive photon detectors that surrounds the intersection region everywhere except within the beams. Calculate an approximate upper bound on the diameter of a photon.

2. Define an interference phenomenon…
   (a) … in terms of sharp and non-sharp observables;
   (b) … in terms of parallel universes;
   (c) Combining your answers to (a) and (b), define an interference phenomenon as a computation.

3. In which of the single-photon experiments (a) to (e) illustrated here can 'which path is the photon on?' be well described by a Boolean observable? In cases where it cannot, how many eigenvalues would an observable describing which path the photon is on have to have?



4. Verify the following properties of the Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$:

   (a) Every $2 \times 2$ Hermitian matrix can be expressed as a linear combination, with real coefficients, of the Pauli matrices and the unit matrix.
   (b) The square of each Pauli matrix is the unit matrix. (And deduce that each Pauli matrix has eigenvalues $\pm 1$.)
   (c) $\sigma_x \sigma_y = -\sigma_y \sigma_x = i\sigma_z$ (and its two cyclic permutations).

5. Suppose that three observables $\hat{X}(t)$, $\hat{Y}(t)$, $\hat{Z}(t)$ of a qubit are such that at time $t = 0$ they have the algebra of the three Pauli matrices. Prove that an arbitrary observable of that qubit has the form $a\hat{X}(t) + b\hat{Y}(t) + c\hat{Z}(t) + d\hat{1}$, where the coefficients *a*, *b*, *c* and *d* are real and do not change with time.

6. Using the observables of example 5, suppose that in a given state, $\langle \hat{Z}(0) \rangle = 1$. Prove that $\langle \hat{X}(0) \rangle = \langle \hat{Y}(0) \rangle = 0$.

7. The qubit of examples 5 and 6 passes through a beam splitter between times 0 and 1. Given that the law of motion for a qubit passing through a beam splitter is

$$\begin{aligned} \hat{X}(t+1) &= -\hat{Z}(t) \\ \hat{Y}(t+1) &= \hat{Y}(t) \\ \hat{Z}(t+1) &= \hat{X}(t) \end{aligned}$$

   (a) Show that $\hat{Z}$ is no longer sharp at time 1.
   (b) Show that $\hat{X}$ is sharp at time 1 and find its value.
   (c) Is $\hat{Y}$ sharp at time 1?

8. The qubit of examples 5-7 then passes through a ***not*** gate (i.e. a mirror on each path, swapping the two directions of motion) between times 1 and 2, and then through another beam splitter between times 2 and 3. The law of motion of a qubit passing through the ***not*** gate is

$$\begin{aligned} \hat{X}(t+1) &= -\hat{X}(t) \\ \hat{Y}(t+1) &= \hat{Y}(t) \\ \hat{Z}(t+1) &= -\hat{Z}(t) \end{aligned}$$

(a) Using your answers to example 7, write down the expectation values of $\hat{X}(2)$, $\hat{Y}(2)$ and $\hat{Z}(2)$.

(b) Write down the expectation values of $\hat{X}(3)$, $\hat{Y}(3)$ and $\hat{Z}(3)$.

(c) Which observable or observables of the qubit remain unchanged throughout this experiment (at least at integer values of the time)?

(d) With respect to which observable or observables of the qubit does this experiment demonstrate an interference phenomenon, and why?

(e) With respect to which observable or observables of the qubit is this *not* an interference phenomenon, and why?

Hints

1. (Suppose that photons are classical impenetrable spheres.) Let $d$ be the diameter of one of these classical photons. How many collisions would occur per second? If they are impenetrable, any collision is likely to deflect them into the detectors.

2. (Define an interference phenomenon)
   (a) The interference phenomenon that I showed in the lecture began and ended with a beam splitter.
   (b) Only one photon at a time participated in the experiment.
   (c) The qubit had different values on the two paths.

3. (Which paths can be described by a Boolean observable.) Count how many paths the photon can be on simultaneously.

4. (Verify properties of the Pauli matrices.)
   (a) Can the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be expressed in that way?
   (b) Just multiply them out. (For the second part, note that the eigenvalues of the square of a matrix are the squares of the eigenvalues of the matrix.)
   (c) Multiply one of them out. Then use algebra.

5. (Real, constant coefficients.) The algebra of the observables is part of the static constitution of a quantum system.

6. ($\langle \hat{Z}(0) \rangle = 1 \quad \Rightarrow \quad \langle \hat{X}(0) \rangle = \langle \hat{Y}(0) \rangle = 0$ .) Consider $\cos\theta \hat{Z}(0) + \sin\theta \hat{X}(0)$ for arbitrary real $\theta$. Prove that it is a Boolean observable, find its eigenvalues and consider the bounds on its expectation value.

7. (After the first beam splitter.) Substitute $t$=0 in the equations of motion and take the expectation value of both sides.

8. (The ***not*** gate and the second beam splitter.)
   (a) Just substitute.
   (b) Just substitute.
   (c) You can see from the equations of motion that $\hat{Y}$ remains unchanged. The unit observable always remains unchanged. What other observables remain unchanged?
   (d) It's interference with respect to the $\hat{Z}$ observable because $\hat{Z}$ starts sharp, becomes un-sharp, and then becomes sharp again. What other observables do this?

(e) It's not interference with respect to $\hat{Y}$, because $\hat{Y}$ remains unchanged. What about $\hat{X}$?

Answers

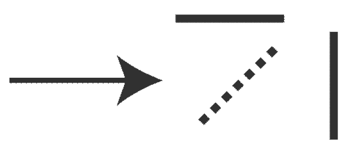
1. (Suppose that photons are classical impenetrable spheres.) Let $d$ be the diameter of one of these spheres, and call the photon rate $r$ and the width of the beam $w$. Then the linear density of photons in the beam is $r/c$ where $c$ is the speed of light, and the area density, as viewed normal to the beam, is $r/cw$. Hence a photon crossing the beam has a probability approximately $rd^2/cw$ of striking another photon. In a time $t = 60\text{s}$, $rt$ photons cross the beam, and if none of them are deflected, the probability that any particular one will be deflected must be much less than $(rt)^{-1}$. In other words, $d \ll r^{-1}\sqrt{cw/t}$, or about $10^{-14}\,\text{m}$.

2. (Define an interference phenomenon)
   (a) A phenomenon in which an easily measurable observable is first sharp, and then becomes un-sharp, and then becomes sharp again.
   (b) A phenomenon in which the observed outcome depends on what has been happening in more than one universe.
   (c) A computation in which a computational variable is first sharp, and then becomes un-sharp, and then becomes sharp again, where the outcome depends on sub-computations that happened in different universes while it was un-sharp.

3. In each case we count the number of paths that the photon can ever be on simultaneously during the experiment. The numbers are:
   (a) 2, so it can be described by a Boolean observable.
   (b) 2, because even after it strikes the beam splitter for a second time, it can still only be travelling leftwards and downwards. So again, it can be described by a Boolean observable.
   (c) 3. At the end of the experiment, it can be travelling leftwards, rightwards and downwards. So it cannot be described by a Boolean observable.
   (d) 2. The mirror is irrelevant here. So it can be described by a Boolean observable.
   (e) 1. A Boolean observable would suffice, but so would the unit observable.

   Note: The following issue was not covered in the Lecture.

   Consider this experiment:



which is the same as case (b) except that the two mirrors are not equidistant from the centre of the beam splitter: the mirror on the horizontal path is farther than, but less than twice as far as, the one on the vertical path. After the photon has struck the beam splitter twice in some universes, it is on three distinct paths, even though all of them are leftwards or downwards. Although the direction of travel is therefore still Boolean to a good approximation, one could not make it sharp again without bringing all three paths together.

4. (Verify properties of the Pauli matrices.)

   (a) Every $2\times 2$ Hermitian matrix has the form $\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}$, where *a, b, c* and *d* are real. But we have

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \frac{a}{2}(\mathrm{I}+\sigma_z),\quad \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = \frac{a}{2}(\mathrm{I}-\sigma_z),\quad \begin{pmatrix} 0 & b+ic \\ b-ic & 0 \end{pmatrix} = \tfrac{1}{2}(b\sigma_x + c\sigma_y),$$

   from which the result follows by inspection.

   (b) Just multiply them out. As for the eigenvalues: since the unit matrix has only the eigenvalue 1, any square root of the unit matrix can only have eigenvalues ±1. A Pauli matrix is not proportional to the unit matrix and must therefore have two distinct eigenvalues values. Hence they must be ±1.

   (c) Express $\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$ in matrix form and perform the matrix multiplications to check it. Then pre-multiply both sides of $\sigma_x\sigma_y = i\sigma_z$ by $\sigma_x$ and use the result of 4(b) ($\sigma_x{}^2 = \mathrm{I}$) to obtain $\sigma_y = i\sigma_x\sigma_z$, which is the first cyclic permutation, and similarly for the remaining equations.

5. (Real, constant coefficients). Let $\hat{A}(t)$ be an arbitrary observable of the qubit, From 4(a) we know that $\hat{A}(0) = a\hat{X}(0) + b\hat{Y}(0) + c\hat{Z}(0) + d\hat{1}$ for some real numbers *a, b, c* and *d*. This is an algebrai relation between observables at time 0. Since the algebra of observables does not change with time, we must also have $\hat{A}(t) = a\hat{X}(t) + b\hat{Y}(t) + c\hat{Z}(t) + d\hat{1}$.

6. ($\langle \hat{Z}(0)\rangle = 1 \;\Rightarrow\; \langle \hat{X}(0)\rangle = \langle \hat{Y}(0)\rangle = 0$.) Consider $\hat{A}_\theta \equiv \cos\theta \hat{Z}(0) + \sin\theta \hat{X}(0)$ for arbitrary real $\theta$. We have $\hat{A}_\theta{}^2 = (\cos^2\theta + \sin^2\theta)\hat{1} + \cos\theta\sin\theta\big(\hat{Z}(0)\hat{X}(0) + \hat{X}(0)\hat{Z}(0)\big) = \hat{1}$. Hence, since $\hat{A}_\theta \neq \hat{1}$, the eigenvalues of $\hat{A}_\theta$ must be ±1. Therefore, from the defining properties of expectation values, $\left|\langle \hat{A}_\theta\rangle\right| \le 1$. Substituting $\langle \hat{Z}(0)\rangle = 1$, we have $\left|\langle \cos\theta + \langle \hat{X}(0)\rangle \sin\theta \rangle\right| \le 1$, or in other words, $\sqrt{1+\langle \hat{X}(0)\rangle^2}\,\sin\big(\theta + \arctan\langle \hat{X}(0)\rangle\big) \le 1$, which cannot hold for all $\theta$ unless $\langle \hat{X}(0)\rangle = 0$. Likewise $\langle \hat{Y}(0)\rangle = 0$.

7. (After the first beam splitter.) From the equations of motion and the results of 6:

   (a) $\langle \hat{Z}(1) \rangle = \langle \hat{X}(0) \rangle = 0$, and so since the eigenvalues of $\hat{Z}$ are ±1, $\hat{Z}(1)$ is not sharp.

   (b) $\langle \hat{X}(1) \rangle = \langle -\hat{Z}(0) \rangle = -\langle \hat{Z}(0) \rangle = -1$. Since the eigenvalues of $\hat{X}$ are ±1, this means $\hat{X}(1)$ must take the value -1 in all universes, so it is sharp.

   (c) $\langle \hat{Y}(1) \rangle = \langle \hat{Y}(0) \rangle = 0$, so $\hat{Y}(1)$ is not sharp.

8. (The ***not*** gate and the second beam splitter.)

   (a) Using the equations of motion for the ***not*** gate, we have

$$
\begin{aligned}
\langle \hat{X}(2) \rangle &= -\langle \hat{X}(1) \rangle = 1 \\
\langle \hat{Y}(2) \rangle &= \langle \hat{Y}(1) \rangle = 0 \\
\langle \hat{Z}(2) \rangle &= -\langle \hat{Z}(1) \rangle = 0
\end{aligned}
$$

   (b) Using the equations of motion for the beam splitter, we have

$$
\begin{aligned}
\langle \hat{X}(3) \rangle &= -\langle \hat{Z}(2) \rangle = 0 \\
\langle \hat{Y}(3) \rangle &= \langle \hat{Y}(2) \rangle = 0 \\
\langle \hat{Z}(3) \rangle &= \langle \hat{X}(2) \rangle = 1
\end{aligned}
$$

   (c) The equations of motion state directly that $\hat{Y}$ remains unchanged, and therefore so does any function of $\hat{Y}$ – which, since $\hat{Y}^2 = \hat{1}$, means any linear combination $a\hat{Y} + b\hat{1}$ with real coefficients *a* and *b*. To make a completely general observable of the qubit, we would add a linear combination of $\hat{X}$ and $\hat{Z}$, but no such combination other than zero remains unchanged, since the ***not*** gate changes both $\hat{X}$ and $\hat{Z}$ to their negatives.

   (d) It's interference with respect to the $\hat{Z}$ observable because $\hat{Z}$ starts sharp, becomes un-sharp, and then becomes sharp again. Evidently any linear combination $a\hat{Z} + b\hat{1}$, for real *a* and *b* with $a \neq 0$ does the same. (If $a = 0$ the observable is always sharp.) But no other non-zero observable of the qubit is sharp at $t = 0$, since $\langle a\hat{X}(0) + b\hat{Y}(0) \rangle = 0$, and so it's interference only with respect to the observables $a\hat{Z} + b\hat{1}$ with $a \neq 0$.