Properties of entanglement assisted channel capacity

Heng Fan
Quantum computation and quantum information project, ERATO,
Japan Science and Technology Corporation,
Daini Hongo White Building 201, Hongo 5-28-3, Bunkyo-ku, Tokyo 133-0033, Japan.

The BSST theorem states that the classical capacity of the entanglement-assisted channel is written as the form [1]

$$C_E(\Phi) = \max_{\rho_A\in\mathcal{H}_{in}} S(\rho_A) + S\left(\Phi(\rho_A)\right) - S\left((\Phi\otimes I)(|\Psi_{AB}\rangle\langle\Psi_{AB}|)\right), \tag{1}$$

where $|\Psi_{AB}\rangle$ is a purification of ρ_A.

Holevo pointed out that there is a relationship between the entanglement-assisted and unassisted capacities [2],

$$C_E(\Phi) \le C(\Phi) + \log d, \tag{2}$$

where d is the dimension of the Hilbert space $\mathcal{H}_{in}$.

If the additivity of the classical capacity holds, we can replace $C(\Phi)$ by one-shot classical capacity $\chi^*(\Phi)$. Since Shor already proved that the classical capacity of the entanglement breaking channel is additive. So, for entanglement breaking channel Φ, we have

$$C_E(\Phi) \le \chi^*(\Phi) + \log d. \tag{3}$$

We can show that a tighter upper bound can be obtained for entanglement breaking channel, that is the following upper bound is correct for entanglement breaking channel,

$$C_E(\Phi) \le \log d. \tag{4}$$

Comparing this relation with the general relation (3), we find that the term $\chi^*(\Phi)$ does not appear here though it is not always zero. So, we show there is an upper bound for $C_E(\Phi)$ when Φ is an entanglement breaking channel. It might be interpreted as, since the channel itself is entanglement-breaking, the prior entanglement may not help much to increase the classical capacity.

Main results: Relationship between entanglement-assisted and one-shot unassisted capacities. Holevo found entanglement-assisted channel capacity is upper bounded by the sum of $\log d$ and the unassisted classical capacity as relation (2) [2]. If the classical channel capacity is additive which is a long-standing conjecture, then we have the inequality

$$C_E(\Phi) \le \chi^*(\Phi) + \log d. \tag{5}$$

For an arbitrary quantum channel Φ, if this relation does not hold, that means $C(\Phi) > \chi^*(\Phi)$, thus the additivity conjecture of classical channel capacity does not hold. So, (5) may provide a criterion to test the additivity problem of classical capacity. However, we can show relation (5) always holds for an arbitrary quantum channel Φ. Thus we eliminate one possibility to find a counter-example for the additivity of classical capacity. We assume that ρ_A have the following pure state decomposition

$$\rho_A = \sum_j q_j|\tilde{\Psi}_A^j\rangle\langle\tilde{\Psi}_A^j|. \tag{6}$$

Using the same technique as that of Shor [5], we define

$$|\tilde{\Psi}_{ABC}\rangle = \sum_j \sqrt{q_j}|\tilde{\Psi}_A^j\rangle|j\rangle_B|j\rangle_C. \tag{7}$$

So, we have

$$(\Phi\otimes I_{BC})(|\tilde{\Psi}_{ABC}\rangle\langle\tilde{\Psi}_{ABC}|) = \sum_{jj'} \sqrt{q_jq_{j'}}\Phi(|\tilde{\Psi}_A^i\rangle\langle\tilde{\Psi}_A^{j'}|)\otimes|j\rangle_B\langle j'|\otimes|j\rangle_C\langle j'|. \tag{8}$$

With the help of the quantum entropy inequality, we obtain

$$S\left((\Phi\otimes I_{BC})(|\tilde{\Psi}_{ABC}\rangle\langle\tilde{\Psi}_{ABC}|)\right) \ge \sum_j q_jS\left(\Phi(|\tilde{\Psi}_A^j\rangle\langle\tilde{\Psi}_A^j|)\right). \tag{9}$$

We know

$$S\left((\Phi\otimes I)(|\Psi_{AB}\rangle\langle\Psi_{AB}|)\right)=S\left((\Phi\otimes I)(|\tilde{\Psi}_{ABC}\rangle\langle\tilde{\Psi}_{ABC}|)\right), \tag{10}$$

where both $|\Psi_{AB}\rangle$ and $|\tilde{\Psi}_{ABC}\rangle$ are purifications of ρ_A. From BSST theorem (1), we have

$$C_E(\Phi)\leq \log d+\chi^*(\Phi). \tag{11}$$

Thus we find a new relationship between the entanglement-assisted and one-shot unassisted capacities. If the additivity of classical capacity holds, this relation is the same as the relation (2). If the additivity does not hold for classical capacity, this relation is tighter than (2).

We can find the following properties of entanglement assisted channel capacity, some already appeared in [3], some are new: 1,Quantum data processing inequalities for entanglement assisted channel capacity,

$$C_E(\mathcal{N}_2\circ\mathcal{N}_1)\leq \min\{C_E(\mathcal{N}_1),C_E(\mathcal{N}_2)\}, \tag{12}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are two discrete memoryless quantum channels such that the input of $\mathcal{N}_2$ is the output of $\mathcal{N}_1$.
2, The convexity of the entanglement assisted channel capacity is written as,

$$C_E(\sum_i p_i\mathcal{N}_i)\leq \sum_i p_iC_E(\mathcal{N}_i). \tag{13}$$

3, Suppose we have different optimal input density operators $\rho_A^i\in\mathcal{H}_{in}$ for quantum channel $\mathcal{N}$, that means the channel capacity $C_E(\mathcal{N})$ can be achieved by every input density operator ρ_A^i, then the average density operator of ρ_A^i with arbitrary probability distribution p_i, $\sum_i p_i=1$, is also the optimal input state. Explicitly, we suppose

$$C_E(\mathcal{N})=S(\rho_A^i)+S(\mathcal{N}(\rho_A^i))-S(\mathcal{N}\otimes I(|\Phi_{AB}^i\rangle\langle\Phi_{AB}^i|)), \tag{14}$$

where $|\Phi_{AB}^i\rangle$ is the purification of ρ_A^i. Then we should have the following result

$$C_E(\mathcal{N})=S(\sum_i p_i\rho_A^i)+S(\mathcal{N}(\sum_i p_i\rho_A^i))-S(\mathcal{N}\otimes I(|\Phi_{AB}\rangle\langle\Phi_{AB}|)), \tag{15}$$

where $|\Phi_{AB}\rangle$ is the purification of $\rho_A=\sum_i p_i\rho_A^i$, i.e., ρ_A is the optimal input state.
4, The entanglement assited classical capacity is additive. See Adami and Cerf PRA.

$$C_E(\mathcal{N}_1\otimes\mathcal{N}_2)=C_E(\mathcal{N}_1)+C_E(\mathcal{N}_2). \tag{16}$$

5, Relationship between entanglement assisted channel capacity with the Holevo bound with the help of results in [4],

$$C_E(\mathcal{N})=\max_{\rho_A\in\mathcal{H}_{in}} S(\rho_A)+C(\{\rho_{A'}^i\})-C(\{\rho_{C'}^i\}). \tag{17}$$

where $C(\{\rho_{A'}^i\})=\max S(\sum_i p_i\rho_{A'}^i)-\sum_i p_iS(\rho_{A'}^i)$ is the Holevo quantity in which the received signals are constrained to lie in a set of quantum states $\rho_{A'}^i$, similarly for $\rho_{C'}^i$ corresponding to the states of the enviornment.

---

[1] C.H. Bennett, P.W.Shor, J.A. Smolin, and A.V. Thapliyal, Phys.Rev.Lett.**83**, 3081 (1999); IEEE Trans. Info. Theory, **48**, 2637 (2002).
[2] A.S.Holevo, J.Math.Phys.**43**, 4326 (2002).
[3] G.Adami and N.Cerf, Phys.Rev.A**56**, 3470 (1997).
[4] B.W.Schumacher, M.D.Westmoreland, Phys.Rev.Lett.**80**, 5695 (1998).
[5] P.Shor, J.Math.Phys.**43**, 4334 (2002).