Quantum walk on the line as an interference phenomenon

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We show that the quantum walk on the line is not a quantum phenomenon in the sense that only wave properties are needed for its understanding.

The quantum random walk (QW for short) concept was first proposed ten years ago by Aharonov *et al.* [1] as the quantum analogue of the classical random walk (RW). As RWs are at the heart of some very efficient classical algorithms, QWs are receiving much attention (see [2] and references therein) in the hope of finding new quantum algorithms that exhibit exponential or polynomial speedup over their classical counterparts.

In a common version of the RW on the line, the "walker" takes one step to the right or to the left randomly depending on the result of tossing a coin. After N steps, the probability of finding the walker at a distance m from the origin is given by the binomial distribution, a Gaussian for large N with variance $\sqrt{N}$. In the QW, the role of the coin is played by a qubit: the quantum walker moves to the right or to the left depending on the internal state of the qubit. After each displacement, the state of the qubit is set to a superposition state by means of a suitable unitary transformation, typically a Hadamard, that plays the role of the toss of the coin in the RW. The Hadamard entangles the position of the particle with the internal state of the coin–qubit. The probability distribution in the quantum case is very different from the classical one: it resembles the Airy function and has a variance that grows linearly with N: the quantum walker walks quadratically faster than the classical one. Moreover, the probability distribution is quite flat and uniform in its central region.

What we demonstrate is that the QW is not quantum in the sense that it can be understood as a classical wave phenomenon. We do this in two ways: (i) we show that the evolution of the walker on the line can be understood by analogy to the propagation of a pulse in a dispersive medium with third order dispersion; and (ii), we devise optical implementations of the QW in an optical cavity.

As for the first approach, it is based on the derivation of the continuous limit of the coined QW on the line. We demonstrate that the probability amplitude for the walker be found at position ξ at time τ is approximately governed by a differential equation of the form

$$\frac{\partial}{\partial\tau}A^{\pm}(\xi,\tau)=\mp\frac{1}{\sqrt{2}}\left[\frac{\partial}{\partial\xi}+\frac{1}{12}\frac{\partial^3}{\partial\xi^3}\right]A^{\pm}(\xi,\tau), \qquad (1)$$

where $A^{\pm}(\xi,\tau)$ are two fields that relate to the probability amplitudes of being at position m at time n, $a_{m,n}$, through $a_{m,n}=A^{+}_{m,n}+(-1)^n A^{-}_{m,n}$, this relation holding separately for the two possible states of the coin. The solution of this equation can be expressed in terms of the Airy function. Eq.(1) describes the propagation of a light pulse in a dispersive medium with third order group velocity dispersion. Thus light pulses can implement the continuous version of the QW by simple propagation.

As for the optical implementations in a cavity, we note the simplest one: consider a ring cavity that contains an electrooptic modulator (EOM) and a half–wave plate (HWP) with its optic axis forming an angle $\pi/8$ with respect to the x axis. Consider a linearly polarized pulse, with the field oscillating at an angle $\pi/4$, injected in the cavity. The EOM increases (decreases) in a cavity free–spectral range the frequency of the $x(y)$ polarization component of the field. Then the light passes through the HWP which performs a Hadamard transformation. The process is then repeated every round trip. This is nothing but the QW on the line: in this case the walker is the field frequency and the light polarization plays the role of the coin. We note that this is not only possible but has indeed been actually implemented by Bouwmeester *et al.* [3], in the context of the optical Galton board, without the authors explicitly making this link to QWs.

The above results suggest that the QW on the line can be understood entirely in terms of interference and can be implemented classically.

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