Comparison of two quantum states by complementary measurements

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Introduction – Mathematical formulations of all the fundamental physical theories are based on the concept of an abstract space. The structure of the space and the theories is characterized in terms of its metric. For example, the Minkowski metric defines the mathematical structure of the special theory of relativity and the Rieman metric defines the structure of the general theory of relativity. What is a natural measure of distance for quantum theory? In particular, two metrics have been applied to the wide realm of quantum information processing: Hilbert-Schmidt distance [1] and Bures metric which corresponds to fidelity [2].

A quantum state is a representation of our knowledge on individual outcomes in future experiments. We can, then intuitively, say that the difference between this knowledge for two quantum states measures how much the two states are *close to each other* with respect to the future predictions. According to Bohr's remark that "...phenomena under different experimental conditions, must be termed complementary in the sense that each is well defined and that together they exhaust all definable knowledge about the object concerned.", the closeness of two quantum states may be defined with regard to a complete set of mutually complementary measurements. We require that such a measure of closeness between two states is invariant under the specific choice of a complete set of mutually complementary measurements.

Definition of operational measure of closeness – Consider two quantum systems of the d-dimensional Hilbert space. In order to indicate how close their density operators ρ_1 and ρ_2 are to each other, we consider a complete set of mutually complementary measurements $M = \{m_\alpha\}$ which are nondegenerate and orthogonal. Consider a measuring device set up with the observable for measurement m_α and let $\{\hat{m}_{\alpha,i}\}$ be the set of the eigen operators and $\{p_{\alpha,i} = \mathrm{Tr}(\hat{m}_{\alpha,i}\hat{\rho})\}$ be the set of probabilities corresponding to the outcomes for a given density operator $\hat{\rho}$. The measurement is performed independently and equivalently for each quantum system and its probability vector is denoted by $\vec{p}_\alpha(S)$ for system S. The distance of the two probability vectors, $\vec{p}_\alpha(1)$ and $\vec{p}_\alpha(2)$, is defined as,

$$D_\alpha(\hat{\rho}_1, \hat{\rho}_2) = |\vec{p}_\alpha(1) - \vec{p}_\alpha(2)|^2, \tag{1}$$

where $|\cdot|$ is a vector norm. The distance D_α is called a single operational distance for measurement m_α belonging to the complete set of mutually complementary measurements. Intuitively, in mutually complementary measurements, one complete knowledge implies maximal uncertainty about the others. The total operational distance may be defined by summing single operational distances over the complete set of complementary measurements:

$$D_{total}(\hat{\rho}_1, \hat{\rho}_2) = \sum_\alpha D_\alpha(\hat{\rho}_1, \hat{\rho}_2). \tag{2}$$

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Equivalence to Hilbert-Schmidt distance – The present geometric approach leads to one of our main results that the total operational distance is equivalent to the Hilbert-Schmidt distance, *i.e.*,

$$D_{total}(\hat{\rho}_1, \hat{\rho}_2) = ||\hat{\rho}_1 - \hat{\rho}_2||^2, \tag{3}$$

where $||\cdot||$ is a Hilbert-Schmidt norm. We remark some properties of the total distance D_{total}: a) the total distance is invariant to the specific choice of a complete set of complementary measurements, b) the total distance is equal to the Hilbert-Schmidt distance of the two operators $\hat{\rho}_1$ and $\hat{\rho}_2$ in the Hilbert-Schmidt space $\mathcal{B}$, and c) the total distance is bounded such that $0 \leq D_{total} \leq 2$.

Information distance – Brukner and Zeilinger [3] introduced the total information content of a quantum system in the density operator $\hat{\rho}$. Their measure can be written in the form of the operational distance as

$$I(\hat{\rho}) = N||\hat{\rho} - \hat{\rho}_r||^2 \tag{4}$$

where N is a normalization factor and $\hat{\rho}_r = \frac{1}{d}\mathbb{1}$ is a completely random state. Compared to the operational distance (3), the total information content $I(\hat{\rho})$ indicates the distance of the quantum state $\hat{\rho}$ from the completely random state $\hat{\rho}_r$. The more information a density operator $\hat{\rho}$ contains, the further it is away from $\hat{\rho}_r$. This find enables to interpret the total operational distance as an *information distance* between two quantum states.

Comparison with fidelity – The fidelity, $F = |\langle\psi|\phi\rangle|^2$, is the transition probability between two pure states, $|\psi\rangle$ and $|\phi\rangle$. When the fidelity is extended to incorporate mixed states as $F = (\mathrm{Tr}\sqrt{\sqrt{\hat{\rho}_1}\hat{\rho}_2\sqrt{\hat{\rho}_1}})^2$, its interpretation becomes vague in an operational perspective. Instead, the fidelity may be indirectly interpreted in terms of statistical distance or "statistical distinguishability" of finding the measurement that optimally resolves neighboring density operators [4].

We compare the fidelity to the operational measure of distance. If the set of test and reference density operators are confined to pure states, the fidelity is equivalent to the total operational distance in the sense that they have the monotonic relation. *The total operational distance is, however, inequivalent to the fidelity as general mixed states are concerned* as the operational distance is not a montonic function of the fidelity but a function of three independent quantities, the fidelity and purities of the states.

Conclusion – In summary, we proposed an operational measure to find how close two quantum states are. This is defined with respect to a complete set of mutually complementary measurements. It was shown that the operational measure is equivalent to the Hilbert-Schmidt distance. The measure provides a remarkable interpretation as an information distance between quantum states. Equipped with the operational distance as a metric, the Hilbert space reflects the information-theoretical foundations of quantum theory.

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