Entangled graphs

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Entanglement, the pure quantum phenomenon first revealed by Schroedinger, has been the topic for discussion and research for many years. Although for bipartite systems entanglement is fairly well understood, for many-particle systems open questions still remain.

In our work, we study the problem of many-particle entanglement. We try to characterize a pure state of many qubits with the help of bipartite entanglement[1] between all possible pairs. With the help of an ***Entangled graph***, which represents the bipartite entanglement properties of a state in a graph, we try to gain a new view on the problem. In the Entangled graph, each particle (qubit) is represented as a vertex and entanglement between a pair of qubits is represented as an edge between two vertices. These edges are non-oriented and can be weighted by concurrence (or better squared concurrence, i.e. the tangle).

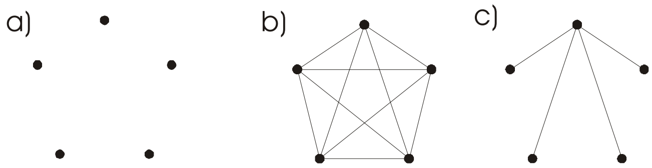

Figure 1. Examples of entangled graphs for five qubits

On the Figure 1, there are some examples of possible Entangled graphs for five qubits. The a) case corresponds to a separable state, or a GHZ type state or any other state where no bipartite entanglement is present. The b) case corresponds to a W state (for example) and the c) case to more complicated states. These cases are of high symmetry and it is easy to find representing pure states for them. However, for five qubits there are of course many other possible graphs (not mentioning systems of more qubits, where the number of combinations is naturally even larger).

In the papers [1,2] we have shown that for every possible graph with N vertices there exists a pure state of N qubits such that the graph represents this state[2]. Moreover, we are able to construct this state explicitly using only a small subspace of available Hilbert space. The whole space is of the dimension 2^N, but we use a subspace only of a dimension proportional to the number of entangled pairs in the system. And this is in the worst case proportional to N^2.

In the paper [3] we enhanced the Entangled graph by introducing weights to the edges. In this case, as it is clear from the CKW-inequalities [4], definitely not for every graph there will be a realization among pure or mixed states. Still, by limiting the maximal allowed concurrence on every edge of the graph, we proved the existence of a state for every such graph. Furthermore, we found an efficient[3] logical network to prepare such states by using only elementary gates such as C-Not, controlled rotations etc.

The second part of our work is devoted to classical correlations. As we know, two particles in a mixed state can be entangled, classically correlated or not correlated at all. Therefore we introduce a

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[1] We use the concurrence as the measure of entanglement

[2] In a sense, that iff a pair of vertices is connected with an edge, the reduced density matrix of corresponding two qubits is entangled

[3] By efficient we understand using in worst case polynomial number of elemetary gates with respect to the number of qubits in the system

new type of edge to our graphs, the classical correlation edge (broken line). On the Figure 2. there are all possible graphs for three qubits with both entanglement and classical correlation edges.

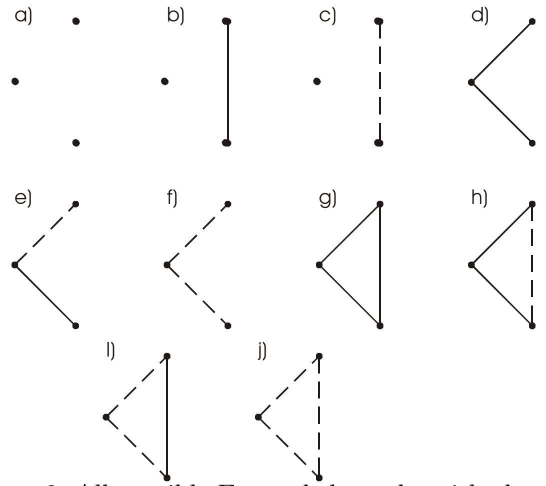

Figure 2. All possible Entangled graphs with classical correlations for three qubits.

In comparison to the case with only one type of legs, the number of possible cases increased from four to ten. The same happens for more qubits; the number of possible graphs grows significantly.

Still we have proved in [5] that one is able to introduce for each graph a mixed state, which is characterized by that graph. The reduced density operator for each qubit is dependent only on the number of particles in the system N and independent of the number of entangled or correlated pairs in the system or a specific configuration. So we are able to prepare a state, where each qubit separately will carry no information, but each pair of qubits will be entangled, correlated or factorized according to a prior request.

On the other hand we can show that not for all graphs a pure state exists. That means, that (for the case of three qubits) there is no pure state, which would be characterized by the graphs c), d), e), and f) according to the Figure 2. One can generalize this fact to bigger number of qubits as a theorem, that only such graphs have representation among pure states, where all edges are either

- Single entanglement edges not connected with the rest of the graph (as the example b))

or

- Part of a closed ring of at least three edges, independently on the type of the edges.

This fact is very interesting, since it shows that every correlation on a pair of qubits in the system, arbitrarily small, can for pure states imply correlation on other pair(s) of the system.

Understanding classical and quantum correlations on systems with many qubits has several assets. It leads to discovering of states that could be used for effective and controlled quantum communication in the system. Possibly, one could come to new ideas for defining and understanding of many-partite entanglement via graphs.

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