The entanglement of indistinguishable particles shared between two parties

H. Wiseman[1], J.A. Vaccaro[1,2]
[1]Centre for Quantum Dynamics, School of Science, Griffith University, Brisbane, 4111 Australia.
[2]Department of Physical Sciences, University of Hertfordshire, College Lane, Hatfield AL10 9AB, U.K.

Entanglement lies at the heart of quantum mechanics, and is profoundly important in quantum information theory. The entanglement of distinguishable particles is an active research area, and recently attention has turned to the entanglement and correlation between indistinguishable particles. In particular two, quite distinct, quantities have been introduced. The first, as championed by Zanardi, we call entanglement of modes, and the second, following Paskauskas and You and others, we call the quantum correlation between two particles [1]. The entanglement of modes E_M is simply determined by calculating the von Neumann entropy of the reduced density operator of one of the parties in the Fock representation. However, this measure overestimates the amount of entanglement between particles. For example, a single particle shared between two parties in the state $|1\rangle|0\rangle+|0\rangle|1\rangle$ has an E_M value of 1 ebit whereas there can be no entanglement of particles for a single particle. In the case of quantum correlations, there is no notion of the location of the particles. The quantum correlation measure is given in terms of the von Neumann entropy of the single particle density matrix, which makes no reference of the sharing of particles between parties at separated sites.

In contrast, we introduce a new concept, the entanglement of indistinguishable particles E_P shared between two separated parties. Our definition of E_P is operationally based in that E_P quantifies the amount of entanglement that can be transferred to another system using local operations (LO), and thus it quantifies the amount of entanglement that can be used as a resource. We consider only pure states $|\psi_{AB}\rangle$ for the particles shared between two parties, Alice and Bob. For genuine particles (such as electrons or Hydrogen atoms), as opposed to gauge bosons (such as photons), there is a conservation law that rules out the creation of superpositions of different number eigenstates. Thus we assume that the state $|\psi_{AB}\rangle$ of the indistinguishable particles contains exactly N particles.

To fully use the entanglement they share, Alice and Bob must be able to arbitrarily measure and manipulate their local systems within their respective Hilbert spaces. Unless Alice's (and hence Bob's) state happens to have a definite particle number, this will mean violating the law of conservation of particle number. To be specific, say that in addition to all of the indistinguishable particles which Alice and Bob may use in the experiment, their quantum state $|\psi_{AB}\rangle$ includes a conventional quantum register each, initially in a product state. The *operational definition* of E_P is the maximal amount of entanglement which Alice and Bob can produce between their quantum registers by local operations. Since the registers of Alice and Bob consist of distinguishable qubits, this entanglement can be computed by the standard measure. As a consequence of the particle number superselection rule, this entanglement will be given by

$$E_P(|\psi_{AB}\rangle) \equiv \sum_{n=0}^{N} P_n \, S[\mathrm{tr}_A \, \rho_{AB}(n)] \tag{1}$$

where $\mathrm{tr}_A(.)$ is the partial trace over Alice's state space and $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy. The density operator $\rho_{AB}(n)$ is given by $\rho_{AB}(n) = |\psi_{AB}(n)\rangle\langle\psi_{AB}(n)|$ where

$$|\psi_{AB}(n)\rangle \equiv \Pi_n |\psi_{AB}\rangle / \sqrt{P_n}$$

is the projection of the state $|\psi_{AB}\rangle$ by the projection operator Π_n onto the subspace in which there are n particles at Alice's site (and $N-n$ at Bob's). P_n is the probability $\langle\psi_{AB}|\Pi_n|\psi_{AB}\rangle$.

To see this explicitly, imagine Alice and Bob perform the optimal LO protocol to change $|\psi_{AB}\rangle$ into the state $|\psi'_{AB}\rangle$ in which their registers have entanglement E_P. Alice could now measure her local particle number, and this would not affect E_P. But since the LO must conserve local particle number, this would be true even if Alice were to measure the particle number at her site *before* applying the LO protocol. This measurement would collapse the state $|\psi_{AB}\rangle$ into the state $|\psi_{AB}(n)\rangle$ with probability $P_n=\langle\psi_{AB}|\Pi_n|\psi_{AB}\rangle$ where n is the measurement result. Now since this is a state of definite local particle number for both parties, there are no conservation laws that prevent local unitaries from transferring all of its entanglement, $S[\mathrm{tr}_A(|\psi_{AB}(n)\rangle\langle\psi_{AB}(n)|)]$, to the quantum registers. To obtain E_P as defined above, one simply averages over the result n.

We show that (at least for bosons) our criterion for entanglement is stronger than both previous criteria, and even their conjunction. We prove a number of properties of our measure, and illustrate it with several examples.

[1] P. Zanardi, Phys. Rev. A **65**, 042101 (2002); R. Paskauskas and L. You, Phys. Rev. A **64**, 042310 (2001).
[2] H.M. Wiseman and J.A. Vaccaro, quant-ph/0210002.