Simple criteria for projective measurements with linear optics

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We derive a set of criteria to decide whether a given projection measurement can be perfectly implemented solely by means of linear optics. The derivation can be adapted to various detection methods, including photon counting and homodyne detection. These criteria enable one to obtain easily No-Go theorems for the perfect distinguishability of orthogonal quantum states with linear optics including the use of auxiliary photons and conditional dynamics.

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One of the goals of quantum communication is to provide a quantum solution to an otherwise unsolvable classical communication problem. An example is quantum key distribution [1, 2]. Part of these solutions may be sub-protocols which run entirely on the quantum level. An example for this is quantum teleportation [3]. As for the implementation of quantum communication protocols, light pulses are the optimal choice, traveling at high speed through an optical fiber and allowing for an efficient broadband encoding of the information.

Joint orthogonal *projection measurements* are an essential tool in quantum communication. The most prominent example is the Bell measurement that is used e.g. in quantum teleportation. The canonical way to perform these measurements optically relies on signal interaction. In discrete-variable implementations based on single photons, however, the required strong nonlinear optical interactions are hard to obtain. Alternatively, it is a promising approach to replace interaction by interference, readily available via *linear optics*, and by feedback after detection. There are important cases, however, where linear optics is not sufficient to enable perfect projective measurements. For instance, a complete measurement in the qubit polarization Bell basis is not possible within the framework of linear optics including beam splitters, phase shifters, auxiliary photons and conditional dynamics utilizing photon counting [4]. Asymptotically, however, the perfect projection onto an orthogonal basis can always be approached via linear optics, though expensive resources such as non-trivial entangled states of auxiliary photons and conditional dynamics are needed then [5]. An important question is how the efficiency of the projection measurement scales as a function of these resources [6]. In any case, No-Go statements always indicate whenever cheaper resources and less sophisticated tools, such as a fixed array of linear optics, are not sufficient for an arbitrarily good efficiency.

Here we propose a new approach to the problem of projective measurements with linear optics and photon counting. Since orthogonal states remain orthogonal after linear optical mode transformations, the inability of perfectly discriminating orthogonal states is due to the measurements in the Fock basis. In the new approach, we replace the actual detections by a dephasing of the (linearly transformed) signal states (see Fig. 1). In other

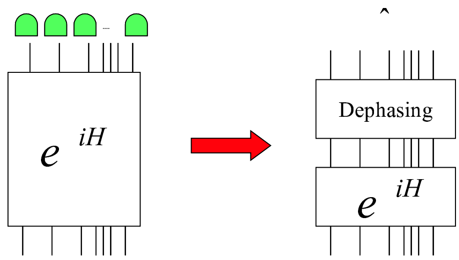

FIG. 1: Dephasing approach to quantum state discrimination via linear optics: the unitary transformation U of the input modes due to linear optics with subsequent photon number detection (on the left) is mimicked by turning the unitarily transformed input states into mixtures of the possible photon number distributions via dephasing (on the right). The output-state density operators after dephasing can be analyzed with respect to their distinguishability.

words, the detection mechanism is mimicked by destroying the coherence of the signal states and turning them into mixtures diagonal in the Fock basis. With the resulting density operators, the distinguishability is then expressible in terms of quantum mechanical states. By considering perfect distinguishability, this yields a hierarchy of simple necessary and sufficient conditions for a complete projection measurement onto an orthogonal basis $\{|\chi_k\rangle\}$,

$$\begin{aligned} \langle\chi_k|\hat{c}_j^\dagger\hat{c}_j|\chi_l\rangle &= 0, \quad \forall j, \\ \langle\chi_k|\hat{c}_j^\dagger\hat{c}_{j'}^\dagger\hat{c}_j\hat{c}_{j'}|\chi_l\rangle &= 0, \quad \forall j, j', \\ \langle\chi_k|\hat{c}_j^\dagger\hat{c}_{j'}^\dagger\hat{c}_{j''}^\dagger\hat{c}_j\hat{c}_{j'}\hat{c}_{j''}\cdots|\chi_l\rangle &= 0, \quad \forall j, j', j'', \\ \vdots \quad &= \quad \vdots \quad \forall k \neq l \end{aligned} \tag{1}$$

where $\hat{c}_j$ are the components of the vector of the output annihilation operators, $\vec{c} = U\vec{a}$, with a unitary matrix U. The hierarchy breaks off for higher-order terms if the number of photons in the states $\{|\chi_k\rangle\}$ is bounded. Hence, for finite photon number, we end up having a finite hierarchy of necessary and sufficient conditions for perfect projective measurements. The states of an orthogonal set $\{|\chi_k\rangle\}$ are perfectly distinguishable via a

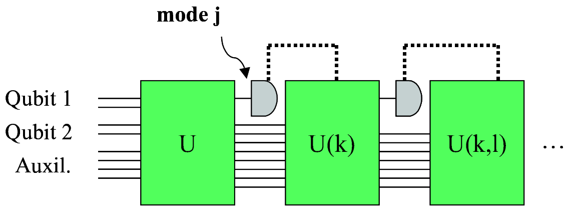

FIG. 2: Employing conditional dynamics and auxiliary photons, for instance, to distinguish two-qubit states: after the first linear optical transformation U of all modes including the auxiliary modes, detect one mode j and apply further transformations to the remaining modes depending on the mesaurement result for mode j.

fixed array of linear optics represented by $\vec{c} = U\vec{a}$, iff these conditions hold for any $k \neq l$.

The subset of conditions referring only to a particular mode operator $\hat{c}_j$ represents necessary conditions for perfect discrimination based on *conditional dynamics* after detecting that mode j (see Fig. 2). They are given by

$$\langle \chi_k | \left(\hat{c}_j^\dagger\right)^n (\hat{c}_j)^n | \chi_l \rangle = 0 \,, \qquad \forall n \geq 1, \quad \forall k \neq l \quad (2)$$

Moreover, one can show that for typical signal states with fixed photon number, adding *auxiliary photons* (see Fig. 2) cannot provide perfect distinguishability, if there is no first-order ($n = 1$) solution of Eq. (2) without them.

The proof of the No-Go theorem for the four qubit Bell states [4] becomes very simple using only the first-order condition of Eq. (2). It can be easily seen that this condition has only trivial solutions, which proves the No-Go theorem for the Bell states including auxiliary photons and conditional dynamics. A similar No-Go theorem for a set of nine separable two-qutrit states [9] can be also easily reproduced using the first-order condition of Eq. (2). Furthermore, new No-Go results can be found, for instance, for an orthogonal set of four arbitrary non-maximally entangled two-qubit states. In addition, the classical distributions of the totally dephased states may be analyzed with respect to their distinguishability in order to make quantitative statements beyond the No-Go theorems. Finally, the dephasing method is applicable to other kinds of measurements too. One may also consider, for example, homodyne detections, i.e., measurements in a continuous-variable basis. The resulting criteria in terms of the output quadratures of a linear-optics array are satisfied, for instance, for the continuous-variable Bell states with a simple 50 : 50 beam splitter. In fact, in the case of a continuous-variable Bell measurement, no No-Go theorem exists and arbitrarily good state discrimination is achievable using a 50 : 50 beam splitter [10].

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